Optimal Ternary Constant-Weight Codes With Weight 4 and Distance 5

Hui Zhang, Xiande Zhang, and Gennian Ge

***Abstract*—Constant-weight codes (CWCs) play an important role in coding theory. The problem of determining the sizes for optimal ternary CWCs with length n, weight 4, and minimum Hamming distance 5 ($(n,5,4)_3$ code) has been settled for all positive integers $n \le 10$ or $n > 10$ and $n \equiv 1 \pmod 3$ with $n \in \{13, 52, 58\}$ undetermined. In this paper, we investigate the problem of constructing optimal $(n,5,4)_3$ codes for all lengths n with the tool of group divisible codes. We determine the size of an optimal $(n,5,4)_3$ code for each integer $n \ge 4$ leaving the lengths $n \in \{12, 13, 21, 27, 33, 39, 45, 52\}$ unsolved.**

***Index Terms*—Constant-weight codes (CWCs), group divisible codes (GDCs), ternary codes.**

I. Introduction

Constant-weight codes (CWCs) play an important role in coding theory [26]. Binary CWCs have been extensively studied by many authors with the focus of attention on the function $A(n,d,w)$, which denotes the maximum cardinality of a binary code of length n, minimum Hamming distance d, and constant weight w.

Nonbinary CWCs have not received the same amount of attention, but there have been a number of papers dealing with this topic, see for example [3]–[5], [7], [11], [12], [25], [29], [32]–[36], and [40], due to several important applications requiring nonbinary alphabets, such as the coding for bandwidth-efficient channels [9] and the design of oligonucleotide sequences for DNA computing [24], [27].

In [29], some methods for providing upper and lower bounds on the maximum cardinality $A_3(n,d,w)$ of a ternary code with length n, minimum Hamming distance d, and constant weight w were presented, and a table of exact values or bounds in the range $n \le 10$ was also given. We list the exact values of $A_3(n,5,4)$ for codes with length n no greater than 10 in Table I.

Manuscript received December 02, 2010; revised June 17, 2011; accepted November 22, 2011. Date of publication January 5, 2012; date of current version April 17, 2012. The work of H. Zhang was supported by the Scholarship Award for Excellent Doctoral Student granted by the China Ministry of Education. The work of G. Ge was supported in part by the National Outstanding Youth Science Foundation of China under Grant 10825103, in part by the National Natural Science Foundation of China under Grant 61171198, the Specialized Research Fund for the Doctoral Program of Higher Education, and in part by the Zhejiang Provincial Natural Science Foundation of China under Grant D7080064.

The authors are with the Department of Mathematics, Zhejiang University, Hangzhou 310027, Zhejiang, China (e-mail: hzhangzju@126.com; xdzhangzju@163.com; gnge@zju.edu.cn).

Communicated by M. Blaum, Associate Editor for Coding Theory.

Digital Object Identifier 10.1109/TIT.2011.2179412

TABLE I
Bounds on $A_3(n,5,4)$ for $n \le 10$

n	4	5	6	7	8	9	10
$A_3(n,5,4)$	1	2	4	7	13	19	30

Generalized Steiner systems $\mathrm{GS}(2,k,n,g)$ were first introduced by Etzion [10] and used to construct optimal CWCs over an alphabet of size $g+1$ with minimum Hamming distance $2k-3$, in which each codeword has length n and weight k. A lot of work has been done on the existence of a $\mathrm{GS}(2,k,n,g)$; see, for example, [2], [8], [10], [13]–[15], [17], [21]–[23], [30], [31], [37], and [39]. In [23], Ji *et al.* proved that a $\mathrm{GS}(2,4,n,2)$ exists if and only if $n \ge 10$ and $n \equiv 1 \pmod 3$ with the possible exceptions $n \in \{13, 52, 58\}$. Equivalently, we have the following lemma.

Lemma 1.1: For any integer $n \ge 10$, $n \equiv 1 \pmod 3$, and $n \notin \{13, 52, 58\}$, there exists an optimal $(n,5,4)_3$ code with $\frac{n(n-1)}{3}$ codewords.

The concept of group divisible codes (GDCs), an analog of group divisible designs (GDDs) in combinatorial design theory [16], was first introduced by Chee *et al.* in [6]. This new class of codes is shown to be useful in recursive constructions of CWCs and constant-composition codes. In [40], GDCs have played a significant role in constructing CWCs with weight 4 and minimum Hamming distance 6. In this paper, we continue to investigate the construction of optimal $(n,5,4)_3$ codes for all lengths n with the tool of GDCs. We determine the size of an optimal $(n,5,4)_3$ code for each integer $n \ge 4$ leaving eight lengths of n unsolved.

This paper is organized as follows. In Section II, basic notations and results in coding theory and combinatorial design theory are given. In Section III, we shall construct some small GDCs which will be used in the following sections. From Sections IV to VII, we will give the constructions for optimal $(n,5,4)_3$ codes case by case. A brief conclusion will be given in the last section.

II. Preliminaries

For integers $m \le n$, the set $\{m, m+1, \ldots, n\}$ is denoted by $[m,n]$. The ring of integers modulo n, $\mathbb{Z}/n\mathbb{Z}$, is denoted by $\mathbb{Z}_n$, and the set of nonnegative integers is denoted by $\mathbb{Z}_{\ge 0}$. For any two sets X and Y, $X \times Y$ denotes the Cartesian product, i.e., $X \times Y = \{(x,y) : x \in X, y \in Y\}$.

A. q-Ary CWCs

If X and R are finite sets, R^X denotes the set of vectors of length $|X|$, where each component of a vector $\mathsf{u} \in R^X$ has value in R and is indexed by an element of X, that is, $\mathsf{u} = (\mathsf{u}_x)_{x\in X}$ and $\mathsf{u}_x \in R$ for each $x \in X$.

A q-ary code of length n is a set $\mathcal{C} \subseteq \mathbb{Z}_q^X$ for some X of size n. The elements of $\mathcal{C}$ are called *codewords*. The *Hamming norm* or the *Hamming weight* of a vector $\mathsf{u} \in \mathbb{Z}_q^X$ is defined as $\|\mathsf{u}\| = |\{x \in X : \mathsf{u}_x \neq 0\}|$. The distance induced by this norm is called the *Hamming distance*, denoted by d_H, so that $d_H(\mathsf{u},\mathsf{v}) = \|\mathsf{u}-\mathsf{v}\|$, for $\mathsf{u},\mathsf{v} \in \mathbb{Z}_q^X$. For any vector $\mathsf{u} \in \mathbb{Z}_q^X$, define the *support* of u as $\operatorname{supp}(\mathsf{u}) = \{x \in X : \mathsf{u}_x \neq 0\}$.

A code $\mathcal{C}$ is said to have (minimum Hamming) *distance* d if $d_H(\mathsf{u},\mathsf{v}) \geq d$ for all distinct $\mathsf{u},\mathsf{v} \in \mathcal{C}$. To simplify the presentation, when we talk about the distance of a code, we always mean the minimum Hamming distance of the code. If $\|\mathsf{u}\| = w$ for every codeword $\mathsf{u} \in \mathcal{C}$, then $\mathcal{C}$ is said to be of (constant) *weight* w. A q-ary CWC of length n, distance d, and weight w is denoted as an $(n,d,w)_q$ code. The number of codewords in an $(n,d,w)_q$ code is called the *size* of the code. The maximum size of an $(n,d,w)_q$ code is denoted as $A_q(n,d,w)$, and the $(n,d,w)_q$ codes achieving this size are said to be *optimal*.

The following bound for $A_q(n,d,w)$ has been established by Svanström [34].

Lemma 2.1 ([34]):

$$A_q(n,d,w) \leq \left\lfloor \frac{n(q-1)}{w} A_q(n-1,d,w-1) \right\rfloor .$$

In the rest of this paper, we will use $U(n,3)$ to denote $\left\lfloor \frac{n}{2} \left\lfloor \frac{2(n-1)}{3} \right\rfloor \right\rfloor$. It is well known that $A_q(n,d,w) = 1$ when $d > 2w$; hence, we get the following upper bound.

Lemma 2.2: $A_3(n,5,4) \leq U(n,3)$.

B. Designs

A *set system* is a pair $(X,\mathcal{B})$ such that X is a finite set of *points* and $\mathcal{B}$ is a set of subsets of X, called *blocks*. The *order* of the set system is $|X|$, the number of points.

For a set of nonnegative integers K, a pairwise balanced design ($(v,K,1)$-PBD) is a set system $(X,\mathcal{B})$ of order v, such that $|B| \in K$ for all $B \in \mathcal{B}$, and every pair of distinct elements of X occurs in exactly one block of $\mathcal{B}$. If an element $k \in K$ is "starred" (written $k^\star$), it means that the PBD has exactly one block of size k.

Lemma 2.3 ([28]): There exists a $(v,\{4,w^\star\},1)$-PBD with $v > w$ if and only if $v \geq 3w+1$, and

1) $v \equiv 1$ or $4 \pmod{12}$ and $w \equiv 1$ or $4 \pmod{12}$; or
2) $v \equiv 7$ or $10 \pmod{12}$ and $w \equiv 7$ or $10 \pmod{12}$.

Let $(X,\mathcal{B})$ be a set system and $\mathcal{G} = \{G_1,\ldots,G_u\}$ be a partition of X into subsets, called *groups*. The triple $(X,\mathcal{G},\mathcal{B})$ is a GDD when each pair of elements of X not contained in a group appears in exactly one block, and $|B \cap G| \leq 1$ for all $B \in \mathcal{B}$ and $G \in \mathcal{G}$. Denote a GDD $(X,\mathcal{G},\mathcal{B})$ by K-GDD if $|B| \in K$ for all $B \in \mathcal{B}$. The *type* of the GDD is the multiset $\{|G| : G \in \mathcal{G}\}$. An "exponential" notation is usually used to describe the type: type $g_1^{u_1} g_2^{u_2} \ldots g_t^{u_t}$ denotes u_i occurrences of g_i, $i = 1,2,\ldots,t$.

A $\{k\}$-GDD of type n^k is also called a *transversal design* and denoted by $\mathrm{TD}(k,n)$.

Lemma 2.4 ([1]): Let m be a positive integer. Then

1) a $\mathrm{TD}(4,m)$ exists if $m \notin \{2,6\}$;
2) a $\mathrm{TD}(5,m)$ exists if $m \notin \{2,3,6,10\}$;
3) a $\mathrm{TD}(6,m)$ exists if $m \notin \{2,3,4,6,10,14,18,22\}$;
4) a $\mathrm{TD}(m+1,m)$ exists if m is a prime power.

As stated in [10] and [39], a $(g+1)$-ary (n,d,k) CWC over $\mathbb{Z}_{g+1}$ can be constructed from a given $\{k\}$-GDD of type g^n, $(I_n \times I_g, \{\{i\} \times I_g : i \in I_n\}, \mathcal{B})$, where $I_m = \{1,2,\ldots,m\}$ and d is the distance of the resulting code. For each block $\{(i_1,a_1),(i_2,a_2),\ldots,(i_k,a_k)\} \in \mathcal{B}$, we form a codeword of length n by putting a_j in position i_j, $1 \leq j \leq k$, and zeros elsewhere. A $\{k\}$-GDD of type g^n which forms a code with distance $2k-3$ is called a generalized Steiner system, denoted by $\mathrm{GS}(2,k,n,g)$.

The following definition of frame generalized Steiner system was first introduced by Ji *et al.* in [23] to construct generalized Steiner systems.

Let $n = h_1u_1 + \cdots + h_tu_t$ and $\mathcal{P}$ be a partition of I_n into u_i subsets of size h_i, $i = 1,2,\ldots,t$. Suppose $(I_n \times I_g, \{P \times I_g : P \in \mathcal{P}\}, \mathcal{B})$ is a $\{k\}$-GDD of type $(h_1g)^{u_1}\ldots(h_tg)^{u_t}$. For each block $\{(i_1,a_1),(i_2,a_2),\ldots,(i_k,a_k)\} \in \mathcal{B}$, we obtain a codeword of length n by putting a_j in position i_j, $1 \leq j \leq k$, and zeros elsewhere. If the resulting code has distance $2k-3$, we call such a $\{k\}$-GDD a frame generalized Steiner system (or shortly a *frame*) of *type* $h_1^{u_1}\ldots h_t^{u_t}$, and denote it by frame-$\mathrm{GS}(2,k,(h_1^{u_1}\ldots h_t^{u_t}),g)$. It is clear that a $\mathrm{GS}(2,k,n,g)$ is also a frame-$\mathrm{GS}(2,k,(1^n),g)$.

C. Group Divisible Codes

Given $\mathsf{u} \in \mathbb{Z}_q^X$ and $Y \subseteq X$, the constriction of u to Y, written $\mathsf{u}|_Y$, is the vector $\mathsf{v} \in \mathbb{Z}_q^Y$ such that $\mathsf{v} = (\mathsf{u}_x)_{x\in Y}$.

Conversely, given $\mathsf{v} \in \mathbb{Z}_q^Y$ and $Y \subseteq X$, the extension of v to X, written $\mathsf{v}|^X$, is the vector $\mathsf{u} \in \mathbb{Z}_q^X$ such that

$$\mathsf{u}_x = \begin{cases} \mathsf{v}_x, & \text{if } x \in Y; \\ 0, & \text{if } x \in X \setminus Y. \end{cases}$$

Given a set $\mathcal{C} \subseteq \mathbb{Z}_q^Y$, let $\mathcal{C}|^X = \{\mathsf{v}|^X : \mathsf{v} \in \mathcal{C}\}$.

A q-ary GDC of distance d is a triple $(X,\mathcal{G},\mathcal{C})$, where $\mathcal{G} = \{G_1,\ldots,G_u\}$ is a partition of X with $|X| = n$, and $\mathcal{C} \subseteq \mathbb{Z}_q^X$ is a q-ary code of length n, such that $d_H(\mathsf{u},\mathsf{v}) \geq d$ for all distinct $\mathsf{u},\mathsf{v} \in \mathcal{C}$, and $\|\mathsf{u}|_{G_i}\| \leq 1$ for all $\mathsf{u} \in \mathcal{C}$, $1 \leq i \leq t$. Elements of $\mathcal{G}$ are called *groups*. We denote a GDC $(X,\mathcal{G},\mathcal{C})$ of distance d as w-GDC(d) if $\mathcal{C}$ is of constant weight w. The *type* of a GDC $(X,\mathcal{G},\mathcal{C})$ is the multiset $\{|G| : G \in \mathcal{G}\}$. As in the case of GDDs, the exponential notation is used to describe the type of a GDC. The *size* of a GDC $(X,\mathcal{G},\mathcal{C})$ is $|\mathcal{C}|$. Note that an $(n,d,w)_q$ code of size c is also a q-ary w-GDC(d) of type 1^n and size c.

The aforementioned notations were first introduced by Chee *et al.* in [6]. By the definition of a frame-GS, we have the following lemma.

Lemma 2.5: If there is a frame-$\mathrm{GS}(2,k,(h_1^{u_1}\ldots h_t^{u_t}),g)$ with b blocks, then there is a $(g+1)$-ary k-GDC$(2k-3)$ of type $h_1^{u_1}\ldots h_t^{u_t}$ and size b.

Taking the frame-GSs given in [23], we have the following lemma.

Lemma 2.6: There exist ternary 4-GDC(5)s of type 3^5 and size 60, type $3^5 6^1$ and size 120, type 2^u and size $\frac{4u(u-1)}{3}$ for each $u \in \{7,13\}$, and type 6^u and size $12u(u-1)$ for each $u \in \{4,5\}$.

The direct constructions of a code or a GDC in this paper are always based on the familiar difference method, where a finite group (mostly abelian group $\mathbb{Z}_n$) will be utilized to generate all the codewords of a code or a GDC. Thus, instead of listing all the codewords, we list a set of *base codewords* and generate others by an additive group or perhaps some further automorphisms.

To save space, for each codeword $\mathsf{u} = (\mathsf{u}_x)_{x\in X}$ of a code or a GDC, we always list the set $\mathrm{supp}(\mathsf{u})$ and use a subscript to denote the value of u_x for each $x \in \mathrm{supp}(\mathsf{u})$. The automorphism group employed always acts on the supports of the base codewords with the subscripts fixed, if it is not mentioned.

Example 2.1: Let $X = \mathbb{Z}_{20}$, and $\mathcal{G} = \{\{i, i+10\} : 0 \le i \le 9\}$. Then, $(X, \mathcal{G}, \mathcal{C})$ is a ternary 4-GDC(5) of type 2^{10} and size 120, where $\mathcal{C}$ is the set of all cyclic shifts of the codewords:

$$11000200020000000000, 10020100000000002000,$$
$$10010000000000020200, 10000012100000000000,$$
$$10200000010000001000, 22000000000020020000.$$

Or equivalently, we can say that $\mathcal{C}$ is obtained by developing the elements of $\mathbb{Z}_{20}$ in the supports of following codewords $+1 \pmod{20}$ with the subscripts fixed:

$$(0_1, 1_1, 5_2, 9_2) \quad (0_1, 5_1, 16_2, 3_2) \quad (0_1, 3_1, 15_2, 17_2)$$
$$(0_1, 6_1, 8_1, 7_2) \quad (0_1, 9_1, 16_1, 2_2) \quad (0_2, 1_2, 12_2, 15_2).$$

It is easy to see that this ternary 4-GDC(5) of type 2^{10} is also an optimal $(20, 5, 4)_3$ code since its size reaches the upper bound in Lemma 2.2.

Example 2.2: Let $X = \mathbb{Z}_{16}$, and $\mathcal{G} = \{\{i, i+4, i+8, i+12\} : 0 \le i \le 3\}$. Then, $(X, \mathcal{G}, \mathcal{C})$ is a ternary 4-GDC(5) of type 4^4 and size 64, where $\mathcal{C}$ is obtained by developing the elements of $\mathbb{Z}_{16}$ in the following supports of codewords $+2 \pmod{16}$ with the subscripts fixed:

$$(1_1, 8_1, 14_1, 7_2) \quad (0_1, 5_1, 7_1, 14_2) \quad (0_1, 1_1, 2_1, 3_2)$$
$$(1_2, 3_2, 4_2, 14_2) \quad (1_1, 4_1, 15_2, 6_2) \quad (1_1, 7_1, 2_2, 4_2)$$
$$(0_1, 13_2, 6_2, 7_2) \quad (1_1, 6_1, 11_2, 0_2).$$

Example 2.3: Let $X = \{0, 1, \ldots, 29\}$, and $\mathcal{G} = \{\{i\} : 0 \le i \le 23\} \cup \{\{24, 25, \ldots, 29\}\}$. Then, $(X, \mathcal{G}, \mathcal{C})$ is a ternary 4-GDC(5) of type $1^{24}6^1$ and size 276, where $\mathcal{C}$ is generated from the following codewords, whose supports are developed under the automorphism group $\langle(0\ 2\ 4\ \cdots\ 22)(1\ 3\ 5\ \cdots\ 23)(24\ 25\ 26)(27\ 28\ 29)\rangle$ with the subscripts fixed:

$$(1_2, 2_2, 5_2, 7_2) \quad (0_1, 5_1, 25_1, 20_2) \quad (1_1, 5_2, 15_2, 28_2)$$
$$(0_1, 26_1, 9_2, 1_2) \quad (0_1, 6_1, 21_1, 13_2) \quad (1_1, 8_1, 10_2, 11_2)$$
$$(1_1, 28_1, 0_2, 2_2) \quad (1_1, 11_1, 9_2, 25_2) \quad (1_1, 8_2, 12_2, 19_2)$$
$$(1_1, 6_1, 27_1, 3_2) \quad (1_1, 12_1, 24_1, 7_2) \quad (1_1, 9_1, 22_2, 27_2)$$
$$(0_1, 14_1, 29_1, 5_2) \quad (1_1, 25_1, 4_2, 18_2) \quad (0_1, 16_1, 14_2, 28_2)$$
$$(0_1, 17_2, 8_2, 27_2) \quad (1_1, 29_1, 6_2, 21_2) \quad (0_2, 13_2, 16_2, 26_2)$$
$$(0_1, 23_2, 4_2, 24_2) \quad (1_1, 2_1, 13_2, 20_2) \quad (1_1, 14_1, 16_1, 24_2)$$
$$(0_1, 4_1, 16_2, 10_2) \quad (1_1, 5_1, 22_1, 23_1).$$

The following constructions are simple but useful, which are given in [6].

Construction 2.1 ([6]): (*Fundamental Construction*) Let $d \le 2(w-1)$, $(X, \mathcal{G}, \mathcal{B})$ be a GDD, and $\omega : X \to \mathbb{Z}_{\ge 0}$ be a weight function. For any subset $S \subseteq X$, let $\widehat{S} = \cup_{x\in S}(\{x\} \times \mathbb{Z}_{\omega(x)})$. Suppose that for each $B \in \mathcal{B}$, there exists a q-ary w-GDC(d) $(\widehat{B}, \{\widehat{\{a\}} : a \in B\}, \mathcal{C}_B)$ of type $\{\omega(a) : a \in B\}$ and size c_B. Then, $(\widehat{X}, \{\widehat{G} : G \in \mathcal{G}\}, \cup_{B\in\mathcal{B}}(\mathcal{C}_B|^{\widehat{X}}))$ is a q-ary w-GDC(d) of type $\{\sum_{x\in G}\omega(x) : G \in \mathcal{G}\}$ and size $\sum_{B\in\mathcal{B}} c_B$.

Example 2.4: Take a $\mathrm{TD}(4,5)$ $(X, \mathcal{G}, \mathcal{B})$ from Lemma 2.4 and apply the Fundamental Construction with weight 4 to every point (that is, $\omega(x) = 4$ for any $x \in X$). For each $B \in \mathcal{B}$, there is a ternary 4-GDC(5) of type 4^4 and size 64 by Example 2.2 (we call it an *input* GDC). The result is a ternary 4-GDC(5) of type 20^4 and size 1600.

Construction 2.2 ([6]): (Filling in Groups) Let $d \le 2(w-1)$. Suppose $(X, \mathcal{G}, \mathcal{C})$ is a q-ary w-GDC(d) of size a. Suppose further that for each group $G \in \mathcal{G}$, there exists a $(|G|, d, w)_q$ code $\mathcal{C}_G$ of size c_G. Then, $\mathcal{C} \cup (\cup_{G\in\mathcal{G}}(\mathcal{C}_G|^X))$ is an $(|X|, d, w)_q$ code of size $a + \sum_{G\in\mathcal{G}} c_G$.

Example 2.5: Take the ternary 4-GDC(5) of type 20^4 and size 1600 from Example 2.4. We have an optimal $(20, 5, 4)_3$ code by Example 2.1. Fill in all the groups with this optimal code. The result is an optimal $(80, 5, 4)_3$ code.

Remark: There is an obvious generalization of Construction 2.2 that allows filling in groups with small GDCs to obtain GDCs of larger size. For example, if we fill in all the groups of the ternary 4-GDC(5) of type 20^4 in Example 2.5 with ternary 4-GDC(5)s of type 2^{10} and size 120, the result is a ternary 4-GDC(5) of type 2^{40} and size 2080.

Construction 2.3: (*Inflation*) Let $d \le 2(w-1)$. Suppose $(X, \mathcal{G}, \mathcal{C})$ is a q-ary w-GDC(d) of size c. Let $\widetilde{X} = X \times \mathbb{Z}_m$ and $\widetilde{\mathcal{G}} = \{G \times \mathbb{Z}_m : G \in \mathcal{G}\}$. Suppose further that there is a $\mathrm{TD}(w, m)$, $(\mathrm{supp}(\mathsf{u}) \times \mathbb{Z}_m, \{\{x\} \times \mathbb{Z}_m : x \in \mathrm{supp}(\mathsf{u})\}, \mathcal{B}_{\mathsf{u}})$, for any codeword $\mathsf{u} \in \mathcal{C}$. For each $B \in \mathcal{B}_{\mathsf{u}}$, form a codeword $\mathsf{v}_B \in \mathbb{Z}_q^{\widetilde{X}}$ with $(\mathsf{v}_B)_{(x,i)} = \mathsf{u}_x$ for $(x, i) \in B$ and zero elsewhere. Then, $(\widetilde{X}, \widetilde{\mathcal{G}}, \cup_{\mathsf{u}\in\mathcal{C}} \cup_{B\in\mathcal{B}_{\mathsf{u}}} \mathsf{v}_B)$ is a q-ary w-GDC(d) of type $\{m|G| : G \in \mathcal{G}\}$ and size cm^2.

Example 2.6: There is a $\mathrm{TD}(4,4)$ by Lemma 2.4. Take the ternary 4-GDC(5) of type 6^4 and size 144 from Lemma 2.6. By Inflation, we obtain a ternary 4-GDC(5) of type 24^4 and size 2304. We also call this procedure *inflating* the ternary 4-GDC(5) of type 6^4 by 4.

Construction 2.4 ([6]): (Adjoining y Points) Let $y \in \mathbb{Z}_{\ge 0}$. Suppose $(X, \mathcal{G}, \mathcal{C})$ is a q-ary w-GDC(d) of size c. Let Y be a set of size y disjoint from X and $X' = X \cup Y$. Suppose further that

1) for a fixed group $G_0 \in \mathcal{G}$, there exists a $(|G_0| + y, d, w)_q$ code $\mathcal{C}_{G_0}$ of size c_{G_0};
2) for any group $G \in \mathcal{G} \setminus \{G_0\}$, there exists a q-ary w-GDC(d) $(G \cup Y, \{\{x\} : x \in G\} \cup \{Y\}, \mathcal{C}_G)$ of type $1^{|G|}y^1$ and size c_G.

Then, $(\mathcal{C}|^{X'}) \cup (\mathcal{C}_{G_0}|^{X'}) \cup (\cup_{G\in\mathcal{G}\setminus\{G_0\}}(\mathcal{C}_G|^{X'}))$ is an $(|X| + y, d, w)_q$ code of size $c + c_{G_0} + \sum_{G\in\mathcal{G}\setminus\{G_0\}} c_G$.

TABLE II
TYPE AND SOURCE OF STARTING GDCs IN LEMMA 5.2

u	Type of GDC	Source
$9s+1$ for $s \geq 4$	$18^{s-1}18^1$ for $s \geq 4$	Lemma 4.4
$9s+4$ or $9s+7$ for $s \geq 32$	$18^{s-1}m^1$ for $s \geq 32$ with $m \in \{24, 30\}$	Lemma 4.6
$9s+4$ or $9s+7$ for $s \in [13, 30]$	$18^{s-1}m^1$ for $s \equiv 0 \pmod 3$, $13 \leq s \leq 28$ with $m \in \{24, 30, 42, 48, 60, 66\}$	Lemma 4.7
$49, 61, 85, 97$	$24^s 24^1$ for $s \in \{3, 4, 6, 7\}$	Lemma 4.8
$76, 106, 286$	$30^s 30^1$ for $s \in \{4, 6, 18\}$	Lemma 4.8
$67, 79, 103, 115, 283$	$24^s 36^1$ for $s \in \{4, 5, 7, 8, 22\}$	Lemma 4.8
$58, 70, 94$	$24^s 18^1$ for $s \in \{4, 5, 7\}$	Lemma 4.8
112	$24^8 30^1$	Lemma 4.8
88	$30^5 24^1$	Lemma 4.8

TABLE III
BASE CODEWORDS OF SMALL GDCs OF TYPE 2^u

u	Codewords					
16	$(0_1, 1_1, 3_1, 23_1)$	$(0_1, 6_1, 11_1, 1_2)$	$(0_1, 30_2, 2_2, 15_2)$	$(0_1, 7_1, 11_2, 13_2)$	$(0_1, 18_1, 25_2, 28_2)$	
	$(0_1, 4_1, 17_1, 9_2)$	$(0_1, 17_2, 18_2, 8_2)$	$(0_1, 3_2, 21_2, 29_2)$	$(0_1, 8_1, 20_2, 31_2)$	$(0_1, 19_2, 26_2, 14_2)$	
19	$(0_1, 1_2, 6_2, 10_2)$	$(0_1, 13_1, 29_2, 3_2)$	$(0_1, 2_1, 17_2, 32_2)$	$(0_1, 8_1, 20_2, 26_2)$	$(0_2, 3_2, 17_2, 28_2)$	$(0_1, 24_1, 28_1, 23_2)$
	$(0_1, 6_1, 4_2, 11_2)$	$(0_1, 1_1, 14_2, 22_2)$	$(0_1, 7_1, 29_1, 31_2)$	$(0_1, 9_2, 25_2, 27_2)$	$(0_1, 18_1, 21_1, 33_1)$	$(0_1, 27_1, 34_2, 35_2)$
22	$(0_1, 2_1, 14_1, 1_2)$	$(0_1, 36_2, 40_2, 5_2)$	$(0_1, 5_1, 26_1, 28_2)$	$(0_1, 17_2, 29_2, 32_2)$	$(0_1, 31_1, 42_2, 37_2)$	
	$(0_1, 9_1, 15_1, 4_2)$	$(0_1, 3_1, 12_2, 30_2)$	$(0_1, 7_1, 17_1, 20_2)$	$(0_1, 19_2, 21_2, 35_2)$	$(0_2, 17_2, 23_2, 24_2)$	
	$(0_1, 1_1, 15_2, 25_2)$	$(0_1, 4_1, 28_1, 38_2)$	$(0_1, 7_2, 18_2, 26_2)$	$(0_1, 25_1, 33_1, 41_2)$		
28	$(0_1, 3_1, 26_2, 7_2)$	$(0_1, 43_1, 54_2, 2_2)$	$(0_1, 6_1, 31_2, 38_2)$	$(0_1, 10_1, 13_2, 47_2)$	$(0_1, 18_1, 27_1, 48_1)$	$(0_1, 33_1, 34_1, 42_2)$
	$(0_1, 2_1, 16_2, 19_2)$	$(0_1, 52_2, 6_2, 20_2)$	$(0_1, 7_1, 24_1, 29_2)$	$(0_1, 11_1, 31_1, 10_2)$	$(0_1, 18_2, 43_2, 45_2)$	$(0_1, 39_2, 40_2, 51_2)$
	$(0_1, 41_2, 1_2, 24_2)$	$(0_1, 5_1, 19_1, 49_2)$	$(0_2, 5_2, 35_2, 43_2)$	$(0_1, 15_1, 48_2, 12_2)$	$(0_1, 21_2, 27_2, 36_2)$	$(0_1, 40_1, 44_1, 34_2)$

Example 2.7: Here is an optimal $(30, 5, 4)_3$ code, which is obtained by developing the elements of $\mathbb{Z}_{30}$ in the following supports of codewords $+2 \pmod{30}$ with the subscripts fixed:

$$\begin{array}{lll} (0_1,1_1,3_1,24_1) & (1_1,6_1,8_2,13_2) & (0_1,5_1,11_2,19_2) \\ (1_1,5_1,15_1,28_2) & (0_1,2_1,20_1,23_2) & (0_1,13_1,26_1,22_2) \\ (0_1,23_1,4_2,8_2) & (1_1,7_1,27_2,9_2) & (1_1,11_2,25_2,4_2) \\ (1_1,9_1,12_1,26_2) & (0_1,25_2,27_2,28_2) & (0_1,21_1,12_2,13_2) \\ (0_1,29_2,5_2,9_2) & (0_1,16_1,6_2,17_2) & (1_1,29_2,6_2,20_2) \\ (1_1,2_1,13_1,17_2) & (0_2,10_2,13_2,18_2) & (0_1,22_1,10_2,16_2) \\ (1_1,0_2,2_2,19_2). & & \end{array}$$

Take the ternary 4-GDC(5) of type 24^4 and size 2304 from Example 2.6. Adjoin six ideal points. Fill in the first three groups together with the six extra points with ternary 4-GDC(5)s of type $1^{24}6^1$ and size 276 from Example 2.3, and fill in the last group together with the six extra points with the aforementioned optimal $(30, 5, 4)_3$ code. The result is an optimal $(102, 5, 4)_3$ code.

In the sequel of this paper, we only consider ternary CWCs with weight 4 and distance 5, so we simply denote the ternary 4-GDC(5) as GDC.

III. SOME SMALL GDCs AND OPTIMAL CODES

In this section, we present some small GDCs and optimal codes, which are required in establishing subsequent results.

Lemma 3.1: There exists a GDC of type 2^u and size $\frac{4u(u-1)}{3}$ for each $u \in \{16, 19, 22, 28\}$.

Proof: For each $u \in \{16, 19, 22, 28\}$, let $X_u = \mathbb{Z}_{2u}$, and $\mathcal{G}_u = \{\{i, i+u\} : 0 \leq i \leq u-1\}$. Then, $(X_u, \mathcal{G}_u, \mathcal{C}_u)$ is a GDC of type 2^u and size $\frac{4u(u-1)}{3}$, where $\mathcal{C}_u$ is obtained by developing the elements of $\mathbb{Z}_{2u}$ in the codewords listed in Table III $+1 \pmod{2u}$. ■

Lemma 3.2: There exists a GDC of type 4^7 and size 224.

Proof: Let $X = \mathbb{Z}_{28}$, and $\mathcal{G} = \{\{i, i+7, i+14, i+21\} : 0 \leq i \leq 6\}$. Then, $(X, \mathcal{G}, \mathcal{C})$ is a GDC of type 4^7 and size 224, where $\mathcal{C}$ is obtained by developing the elements of $\mathbb{Z}_{28}$ in the following codewords $+1 \pmod{28}$:

$$\begin{array}{lll} (0_1,17_1,4_2,19_2) & (0_1,12_2,13_2,17_2) & (0_1,6_1,8_1,9_1) \\ (0_1,4_1,10_2,26_2) & (0_1,18_2,24_2,27_2) & (0_1,3_2,5_2,23_2) \\ (0_1,16_1,8_2,25_2) & (0_1,10_1,23_1,11_2). & \end{array}$$

■

Lemma 3.3: There exists a GDC of type 6^u and size $12u(u-1)$ for each $u \in [6, 10] \cup \{19\}$.

Proof: For each $u \in \{6, 8, 9\}$, let $X_u = \mathbb{Z}_{6u}$, and $\mathcal{G}_u = \{\{i, i+u, \ldots, i+5u\} : 0 \leq i \leq u-1\}$. Then, $(X_u, \mathcal{G}_u, \mathcal{C}_u)$ is a GDC of type 6^u and size $12u(u-1)$, where $\mathcal{C}_u$ is obtained by developing the elements of $\mathbb{Z}_{6u}$ in the following codewords $+1 \pmod{6u}$:

$u = 6$:

$$\begin{array}{lll} (0_1,2_2,9_2,10_2) & (0_1,10_1,19_1,21_1) & (0_1,28_1,31_1,29_2) \\ (0_1,5_2,8_2,25_2) & (0_1,13_2,15_2,26_2) & (0_1,32_1,16_2,31_2) \\ (0_1,13_1,29_1,4_2) & (0_1,14_1,17_2,21_2) & (0_1,14_2,19_2,28_2) \\ (0_1,1_1,23_2,33_2) & & \end{array}$$

$u = 8$:

$$\begin{array}{lll} (0_1,2_1,44_1,1_2) & (0_1,10_2,27_2,38_2) & (0_1,28_1,31_1,41_1) \\ (0_1,26_1,27_1,9_2) & (0_1,12_2,18_2,19_2) & (0_1,29_1,34_1,14_2) \\ (0_1,30_1,2_2,23_2) & (0_1,21_2,39_2,44_2) & (0_1,33_1,37_2,46_2) \\ (0_1,3_2,15_2,29_2) & (0_1,22_2,25_2,35_2) & (0_1,39_1,34_2,36_2) \\ (0_1,7_2,11_2,26_2) & (0_1,25_1,36_1,42_2) & \end{array}$$

$u = 9$:

TABLE IV
BASE CODEWORDS OF SOME SMALL GDCS IN LEMMA 3.7

Types	Codewords					
$12^4 18^1$	$(0_1,1_1,3_1,52_2)$	$(0_2,1_2,38_2,53_1)$	$(1_2,7_2,14_2,56_2)$	$(0_1,33_1,39_1,60_2)$	$(0_2,13_2,31_2,60_1)$	$(1_1,23_1,48_1,38_2)$
	$(0_1,6_1,51_1,9_2)$	$(1_1,24_1,57_1,2_2)$	$(0_1,11_1,46_2,53_2)$	$(0_1,37_1,49_1,30_2)$	$(0_2,14_2,23_2,49_1)$	$(1_1,28_1,39_2,55_2)$
	$(1_1,6_1,3_2,54_2)$	$(1_1,4_1,11_1,56_1)$	$(0_1,13_1,14_1,58_1)$	$(0_1,39_2,42_2,64_2)$	$(1_1,14_1,27_2,62_2)$	$(1_1,28_2,35_2,50_2)$
	$(0_1,17_1,2_2,51_2)$	$(1_1,53_1,23_2,8_2)$	$(0_1,15_1,29_1,53_1)$	$(0_1,41_2,31_2,54_2)$	$(1_1,14_2,19_2,64_2)$	$(1_1,30_1,40_1,54_1)$
	$(0_1,26_1,1_2,50_2)$	$(1_1,60_1,6_2,32_2)$	$(0_1,18_2,21_2,48_2)$	$(0_1,59_1,34_2,19_2)$	$(1_1,15_2,46_2,60_2)$	$(1_1,43_2,44_2,58_2)$
	$(0_1,2_1,27_2,61_2)$	$(1_1,61_1,0_2,30_2)$	$(0_1,22_2,43_2,55_2)$	$(0_1,61_1,47_2,33_2)$	$(1_1,18_1,24_2,65_2)$	$(1_1,52_1,11_2,22_2)$
	$(0_1,54_1,7_2,29_2)$	$(1_1,62_1,47_2,4_2)$	$(0_1,23_1,41_1,52_1)$	$(0_1,62_1,10_2,37_2)$	$(1_1,18_2,20_2,61_2)$	$(1_1,63_1,31_2,40_2)$
	$(0_1,5_1,14_2,57_2)$	$(1_1,7_2,26_2,53_2)$	$(0_1,30_1,56_1,35_2)$	$(0_1,64_1,15_2,17_2)$	$(1_1,22_1,12_2,59_2)$	$(1_2,24_2,30_2,60_2)$
$12^5 18^1$	$(0_2,2_2,8_2,77_2)$	$(0_1,8_1,54_1,76_1)$	$(0_1,17_1,19_2,69_2)$	$(0_1,32_1,36_1,39_2)$	$(0_1,60_1,23_2,32_2)$	$(0_1,73_1,51_2,58_2)$
	$(0_1,1_1,14_2,74_2)$	$(0_1,9_2,36_2,63_2)$	$(0_1,23_1,34_2,65_2)$	$(0_1,34_1,52_2,16_2)$	$(0_1,65_1,33_2,47_2)$	$(0_2,12_2,13_2,16_2)$
	$(0_1,2_1,64_1,31_2)$	$(0_1,11_1,77_1,17_2)$	$(0_1,24_2,41_2,71_2)$	$(0_1,38_1,47_1,74_1)$	$(0_1,67_1,37_2,59_2)$	$(0_2,67_1,18_2,41_2)$
	$(0_1,3_1,21_1,75_2)$	$(0_1,12_2,38_2,64_2)$	$(0_1,27_1,48_2,70_2)$	$(0_1,41_1,48_1,49_2)$	$(0_1,69_1,22_2,53_2)$	$(0_1,71_1,54_2,26_2)$
	$(0_1,43_2,4_2,68_2)$	$(0_1,16_1,61_1,44_2)$	$(0_1,31_1,27_2,62_2)$	$(0_1,57_2,46_2,77_2)$		
$12^6 18^1$	$(0_1,2_1,9_1,65_2)$	$(0_1,5_1,32_1,33_1)$	$(0_1,19_1,39_2,79_2)$	$(0_1,26_2,55_2,83_2)$	$(0_1,51_2,58_2,31_2)$	$(0_2,72_1,41_2,55_2)$
	$(0_1,3_1,46_2,7_2)$	$(0_1,76_1,9_2,25_2)$	$(0_1,21_2,32_2,77_2)$	$(0_1,34_2,47_2,72_2)$	$(0_1,55_1,81_1,23_2)$	$(0_2,79_1,67_2,68_2)$
	$(0_1,4_1,84_1,1_2)$	$(0_1,85_1,28_2,3_2)$	$(0_1,22_1,35_2,82_2)$	$(0_1,35_1,64_1,74_1)$	$(0_1,59_2,15_2,80_2)$	$(0_1,25_1,75_1,70_2)$
	$(0_1,10_1,87_1,5_2)$	$(0_1,11_1,10_2,73_2)$	$(0_1,22_2,14_2,88_2)$	$(0_1,38_1,52_1,78_1)$	$(0_1,82_1,52_2,62_2)$	$(0_1,50_2,16_2,86_2)$
	$(0_1,21_1,2_2,17_2)$	$(0_1,13_1,57_2,76_2)$	$(0_1,23_1,61_2,89_2)$	$(0_1,46_1,11_2,85_2)$	$(0_1,88_1,41_2,19_2)$	$(0_2,23_2,26_2,63_2)$
	$(0_1,27_2,29_2,8_2)$	$(0_1,15_1,31_1,64_2)$				
$12^7 18^1$	$(0_1,3_1,86_1,8_2)$	$(0_2,8_2,69_2,90_2)$	$(0_1,19_2,46_2,58_2)$	$(0_1,27_1,68_2,93_2)$	$(0_1,48_2,18_2,92_2)$	$(0_1,93_1,59_2,65_2)$
	$(0_1,10_1,92_1,9_2)$	$(0_2,9_2,59_2,86_2)$	$(0_1,1_1,100_1,82_2)$	$(0_1,29_1,91_1,54_2)$	$(0_1,51_1,57_2,33_2)$	$(0_1,97_1,47_2,80_2)$
	$(0_1,2_1,39_1,84_2)$	$(0_1,11_1,19_1,94_2)$	$(0_1,20_1,24_1,40_2)$	$(0_1,34_2,50_2,86_2)$	$(0_1,52_2,12_2,23_2)$	$(0_2,86_1,17_2,58_2)$
	$(0_1,2_2,55_2,91_2)$	$(0_1,12_1,17_1,44_2)$	$(0_1,22_1,11_2,98_2)$	$(0_1,38_1,53_1,87_1)$	$(0_1,62_2,67_2,15_2)$	$(0_2,92_1,65_2,83_2)$
	$(0_1,48_1,61_1,3_2)$	$(0_1,13_2,17_2,96_2)$	$(0_1,25_1,34_1,29_2)$	$(0_1,40_1,64_2,89_2)$	$(0_1,74_2,36_2,38_2)$	$(0_1,16_1,61_2,101_2)$
	$(0_1,72_2,75_2,1_2)$	$(0_1,18_1,90_1,78_2)$	$(0_1,26_1,32_1,69_2)$	$(0_1,43_1,10_2,30_2)$	$(0_1,84_1,31_2,53_2)$	$(0_1,30_1,101_1,22_2)$
$12^8 18^1$	$(0_1,30_2,5_2,96_2)$	$(0_1,104_1,47_2,9_2)$	$(0_1,20_1,33_1,62_1)$	$(0_1,44_1,96_1,73_2)$	$(0_2,3_2,33_2,113_2)$	$(0_1,27_1,92_2,111_2)$
	$(0_1,35_1,7_2,41_2)$	$(0_1,10_2,11_2,31_2)$	$(0_1,20_2,66_2,77_2)$	$(0_1,51_2,53_2,63_2)$	$(0_1,105_1,14_2,61_2)$	$(0_1,33_2,46_2,105_2)$
	$(0_1,3_1,89_2,15_2)$	$(0_1,12_1,50_1,57_1)$	$(0_1,25_1,83_2,23_2)$	$(0_1,70_1,19_2,26_2)$	$(0_1,110_1,76_2,95_2)$	$(0_1,43_1,60_2,109_2)$
	$(0_1,3_2,34_2,98_2)$	$(0_1,14_1,57_2,85_2)$	$(0_1,2_1,113_1,39_2)$	$(0_1,87_2,35_2,50_2)$	$(0_1,11_1,17_1,112_2)$	$(0_1,49_1,108_1,67_2)$
	$(0_1,59_1,1_2,99_2)$	$(0_1,15_1,84_2,42_2)$	$(0_1,31_1,98_1,59_2)$	$(0_1,99_1,21_2,25_2)$	$(0_1,13_2,82_2,103_2)$	$(0_2,43_2,78_2,106_2)$
	$(0_1,75_2,81_2,2_2)$	$(0_1,19_1,23_1,93_2)$	$(0_1,36_1,41_1,90_2)$	$(0_1,9_1,30_1,106_1)$	$(0_1,18_1,28_1,110_2)$	$(0_1,22_1,44_2,101_2)$
	$(0_1,101_1,55_2,4_2)$	$(0_1,1_1,103_1,79_2)$	$(0_1,36_2,62_2,91_2)$	$(0_2,109_1,9_2,91_2)$		
$12^9 18^1$	$(0_1,24_1,29_1,8_2)$	$(0_1,22_1,25_1,73_1)$	$(0_1,69_1,76_1,71_2)$	$(0_1,13_1,71_1,110_2)$	$(0_1,42_2,86_2,119_2)$	$(0_1,121_1,100_2,3_2)$
	$(0_1,30_1,85_1,5_2)$	$(0_1,2_1,124_1,97_2)$	$(0_1,70_1,37_2,74_2)$	$(0_1,17_1,110_1,55_2)$	$(0_1,48_2,50_2,114_2)$	$(0_1,119_1,30_2,46_2)$
	$(0_1,14_1,24_2,39_2)$	$(0_1,31_1,43_1,47_1)$	$(0_1,8_1,109_1,21_2)$	$(0_1,21_1,115_1,41_2)$	$(0_1,65_2,91_2,112_2)$	$(0_1,125_1,59_2,102_2)$
	$(0_1,15_1,64_2,16_2)$	$(0_1,33_1,89_2,32_2)$	$(0_1,97_1,40_2,62_2)$	$(0_1,28_1,80_2,115_2)$	$(0_1,82_2,96_2,121_2)$	$(0_1,82_1,112_1,104_2)$
	$(0_1,15_2,35_2,70_2)$	$(0_1,34_1,44_1,78_2)$	$(0_2,24_2,25_2,28_2)$	$(0_1,33_2,14_2,117_2)$	$(0_1,85_2,106_2,47_2)$	$(0_1,117_1,93_2,101_2)$
	$(0_1,19_1,76_2,26_2)$	$(0_1,49_1,29_2,61_2)$	$(0_2,30_2,42_2,47_2)$	$(0_1,40_1,46_1,122_2)$	$(0_2,108_1,33_2,85_2)$	$(0_1,107_1,67_2,113_2)$
	$(0_1,19_2,58_2,98_2)$	$(0_1,52_1,23_2,69_2)$	$(0_1,42_1,114_1,11_2)$	$(0_1,41_1,116_1,84_2)$	$(0_2,10_2,41_2,108_2)$	$(0_1,105_2,31_2,125_2)$
	$(0_1,20_1,6_2,116_2)$	$(0_1,53_2,60_2,66_2)$				

$$
\begin{array}{lll}
(0_1,5_2,7_2,8_2) & (0_1,10_1,51_1,34_2) & (0_1,26_2,31_2,42_2) \\
(0_1,1_1,7_1,20_2) & (0_1,10_2,32_2,47_2) & (0_1,30_1,35_1,16_2) \\
(0_1,2_1,17_1,33_1) & (0_1,11_1,25_1,23_2) & (0_1,46_1,22_2,48_2) \\
(0_1,3_2,44_2,50_2) & (0_1,17_2,38_2,46_2) & (0_1,50_1,39_2,51_2) \\
(0_1,4_2,14_2,28_2) & (0_1,20_1,42_1,53_2) & (0_1,26_1,21_2,41_2) \\
(0_1,6_2,25_2,29_2). & &
\end{array}
$$

For $u \in \{7, 10, 19\}$, inflate GDCs of type 2^u (see Example 2.1, Lemmas 2.6 and 3.1) by 3 to obtain the required GDCs. ■

Lemma 3.4: There exists a GDC of type $3^6 6^1$ and size 162.

Proof: Let $X = \{0, 1, \ldots, 23\}$, and $\mathcal{G} = \{\{i, i+6, i+12\} : 0 \le i \le 5\} \cup \{\{18, \ldots, 23\}\}$. Then, $(X, \mathcal{G}, \mathcal{C})$ is a GDC of type $3^6 6^1$ and size 162, where $\mathcal{C}$ is generated from the following codewords, which are developed under the automorphism group $\langle (0\ 2\ 4\ \cdots\ 16)(1\ 3\ 5\ \cdots\ 17)(18\ 19\ 20)\ (21\ 22\ 23) \rangle$:

$$
\begin{array}{lll}
(0_1,1_2,4_2,19_2) & (1_1,3_1,6_1,16_2) & (1_1,12_1,5_2,20_2) \\
(0_1,22_1,7_2,8_2) & (1_1,8_1,6_2,22_2) & (1_2,9_2,14_2,19_1) \\
(0_1,3_1,20_1,2_2) & (0_1,14_1,21_1,9_2) & (0_1,16_1,15_2,22_2) \\
(0_1,3_2,5_2,14_2) & (0_1,8_1,17_1,19_1) & (1_1,10_2,12_2,23_2) \\
(1_1,18_1,2_2,3_2) & (0_2,3_2,10_2,18_2) & (1_1,11_2,15_2,21_2) \\
(1_1,21_1,4_2,8_2) & (1_1,11_1,22_1,9_2) & (1_1,14_1,15_1,19_2).
\end{array}
$$

■

Lemma 3.5: There exists a GDC of type $6^u 3^1$ and size $12u^2$ for each $u \in \{4, 5\}$.

Proof: For each $u \in \{4, 5\}$, let $X_u = \{0, 1, \ldots, 6u+2\}$, and $\mathcal{G}_u = \{\{i, i+u, \ldots, i+5u\} : 0 \le i \le u-1\} \cup \{\{6u, 6u+1, 6u+2\}\}$. Then, $(X_u, \mathcal{G}_u, \mathcal{C}_u)$ is a GDC of type $6^u 3^1$ and size $12u^2$, where $\mathcal{C}_u$ is generated from the following codewords, which are developed under the automorphism group $\langle (0\ 1\ 2\ \cdots\ 6u-1)(6u\ 6u+1\ 6u+2) \rangle$:

$u = 4$:

$$
\begin{array}{lll}
(0_1,2_1,15_1,21_1) & (0_1,17_1,10_2,23_2) & (0_1,2_2,5_2,11_2) \\
(0_1,3_2,13_2,26_2) & (0_1,25_1,21_2,22_2) & (0_1,7_2,9_2,14_2) \\
(0_1,1_1,19_2,25_2) & (0_1,14_1,26_1,15_2). &
\end{array}
$$

$u = 5$:

$$
\begin{array}{lll}
(0_1,4_1,3_2,27_2) & (0_1,1_1,14_2,30_2) & (0_1,9_1,12_1,28_1) \\
(0_1,4_2,6_2,22_2) & (0_1,30_1,2_2,21_2) & (0_1,12_2,16_2,19_2) \\
(0_1,7_1,24_1,1_2) & (0_1,8_1,31_1,26_2) & (0_1,11_2,28_2,31_2) \\
(0_1,8_2,9_2,17_2). & &
\end{array}
$$

■

Lemma 3.6: There exists a GDC of type 12^u and size $48u(u-1)$ for all $u \ge 4$.

Proof: When $u \equiv 0$ or $1 \pmod 4$ and $u \ge 4$, there exists a $(3u+1, \{4\}, 1)$-PBD by Lemma 2.3. Deleting one point from the point set gives a $\{4\}$-GDD of type 3^u. When $u \equiv 2$ or 3 $\pmod 4$ and $u \geq 7$, there exists a $(3u+1, \{4, 7^\star\}, 1)$-PBD by Lemma 2.3. Remove one point not contained in the block of size 7 from the point set to obtain a $\{4, 7^\star\}$-GDD of type 3^u. Hence, we always have a $\{4, 7\}$-GDD of type 3^u for any $u \geq 4$ and $u \neq 6$.

Start from the aforementioned $\{4, 7\}$-GDDs of type 3^u, and apply the Fundamental Construction with weight 4 to obtain GDCs of type 12^u for all $u \geq 4$ and $u \neq 6$. Here, the input GDCs of types 4^4 and 4^7 exist by Example 2.2 and Lemma 3.2.

For $u = 6$, take a $\{5\}$-GDD of type 4^6 (see [19]), and apply the Fundamental Construction with weight 3 to obtain the required GDC. Here, the input GDC of type 3^5 is from Lemma 2.6. ■

Lemma 3.7: There exists a GDC of type $12^u 18^1$ and size $48u(u+2)$ for each $u \in [4, 9]$.

Proof: For each $u \in [4,9]$, let $X_u = \{0, 1, \ldots, 12u + 17\}$, and $\mathcal{G}_u = \{\{i, i+u, \ldots, i+11u\} : 0 \leq i \leq u-1\} \cup \{\{12u, \ldots, 12u+17\}\}$. Then, $(X_u, \mathcal{G}_u, \mathcal{C}_u)$ is a GDC of type $12^u 18^1$ and size $48u(u+2)$, where $\mathcal{C}_u$ is generated from the codewords listed in Table IV, which are developed under the automorphism group G as indicated next.

For $u = 4, G = \langle (0\ 2\ 4\ \cdots\ 46)(1\ 3\ 5\ \cdots\ 47)(48\ 51\ 54\ \cdots\ 63)\ (49\ 52\ 55\ \cdots\ 64)(50\ 53\ 56\ \cdots\ 65)\rangle$.

For $u \in \{5, 7, 8, 9\}$, $G = \langle (0\ 1\ 2\ \cdots\ 12u-1)(12u\ 12u+1\ 12u+2\ \cdots\ 12u+5)\ (12u+6\ 12u+7\ 12u+8\ \cdots\ 12u+17)\rangle$.

For $u = 6$, $G = \langle (0\ 1\ 2\ \cdots\ 71)(72\ 73\ 74)(75\ 76\ 77)(78\ 79\ 80)(81\ 82\ 83)(84\ 85\ 86)(87\ 88\ 89)\rangle$. ■

Lemma 3.8: There exists a GDC of type $1^{12}4^1$ and size 72.

Proof: Let $X = \{0, 1, \ldots, 15\}$, and $\mathcal{G} = \{\{i\} : 0 \leq i \leq 11\} \cup \{\{12, \ldots, 15\}\}$. Then, $(X, \mathcal{G}, \mathcal{C})$ is a GDC of type $1^{12}4^1$ and size 72, where $\mathcal{C}$ is generated from the following codewords, which are developed under the automorphism group $\langle (0\ 4\ 8)(1\ 5\ 9)(2\ 6\ 10)(3\ 7\ 11)(12\ 13\ 14)(15)\rangle$:

$(0_1, 2_1, 7_1, 1_2)$	$(1_1, 2_1, 4_1, 12_2)$	$(3_1, 7_1, 0_2, 14_2)$
$(1_1, 0_2, 2_2, 5_2)$	$(1_1, 4_2, 7_2, 13_2)$	$(0_1, 13_1, 4_2, 11_2)$
$(2_1, 6_2, 7_2, 8_2)$	$(1_1, 6_1, 9_2, 14_2)$	$(0_1, 3_2, 10_2, 13_2)$
$(0_1, 1_1, 15_1, 8_2)$	$(1_2, 4_2, 5_2, 14_2)$	$(1_1, 12_1, 6_2, 10_2)$
$(0_1, 3_1, 6_2, 15_2)$	$(2_1, 14_1, 0_2, 4_2)$	$(1_1, 9_1, 11_1, 13_1)$
$(0_1, 4_1, 12_1, 9_2)$	$(2_1, 3_1, 6_1, 13_1)$	$(1_2, 7_2, 11_2, 14_1)$
$(0_1, 5_1, 11_1, 7_2)$	$(2_1, 5_1, 3_2, 15_2)$	$(2_1, 15_1, 9_2, 11_2)$
$(0_1, 6_1, 2_2, 12_2)$	$(3_1, 12_1, 1_2, 2_2)$	$(3_1, 7_2, 10_2, 12_2)$.

■

We have the following improvement on $A_3(n, 5, 4)$ for $n \equiv 1 \pmod 3$.

Lemma 3.9: $A_3(58, 5, 4) = U(58, 3)$, $A_3(13, 5, 4) \geq U(13, 3) - 4$, and $A_3(52, 5, 4) \geq U(52, 3) - 12$.

Proof: For $n = 58$, the required code is constructed on $\mathbb{Z}_{58}$, and obtained by developing the elements of $\mathbb{Z}_{58}$ in the following codewords $+2 \pmod{58}$:

$(1_1, 3_2, 5_2, 6_2)$	$(1_1, 24_1, 48_2, 4_2)$	$(0_1, 44_2, 19_2, 26_2)$
$(0_1, 4_1, 47_2, 5_2)$	$(1_1, 37_2, 8_2, 17_2)$	$(0_1, 57_2, 36_2, 42_2)$
$(0_1, 6_2, 9_2, 49_2)$	$(1_1, 45_1, 48_1, 2_2)$	$(1_1, 10_2, 20_2, 46_2)$
$(1_1, 2_1, 5_1, 34_1)$	$(1_1, 6_1, 22_1, 29_1)$	$(1_1, 11_1, 18_1, 57_2)$
$(0_1, 19_1, 53_2, 3_2)$	$(1_1, 7_1, 55_2, 32_2)$	$(1_1, 12_1, 27_1, 39_2)$
$(0_1, 1_1, 50_2, 31_2)$	$(0_1, 10_2, 11_2, 35_2)$	$(1_1, 14_1, 19_1, 27_2)$
$(0_1, 21_1, 54_2, 4_2)$	$(0_1, 13_1, 37_2, 23_2)$	$(1_1, 21_1, 23_1, 41_2)$
$(0_1, 2_1, 14_1, 24_1)$	$(0_1, 17_1, 14_2, 16_2)$	$(1_1, 23_2, 29_2, 51_2)$
$(0_1, 2_2, 29_2, 33_2)$	$(0_1, 18_1, 15_2, 25_2)$	$(1_1, 24_2, 40_2, 52_2)$
$(0_1, 6_1, 46_2, 51_2)$	$(0_1, 27_1, 41_2, 22_2)$	$(1_1, 33_2, 38_2, 45_2)$
$(0_1, 8_1, 28_1, 56_2)$	$(0_1, 30_2, 34_2, 17_2)$	$(1_1, 35_1, 50_1, 12_2)$
$(0_1, 8_2, 21_2, 32_2)$	$(0_1, 31_1, 39_1, 52_2)$	$(1_2, 10_2, 27_2, 48_2)$
$(1_1, 13_1, 30_2, 7_2)$	$(0_1, 33_1, 49_1, 18_2)$.	

For $n = 13$, the desired code is constructed on $\{0, 1, 2, \ldots, 12\}$ with 48 codewords listed next:

$(0_1, 7_2, 4_1, 6_1)$	$(10_1, 1_2, 8_1, 6_1)$	$(8_1, 0_1, 11_2, 5_2)$
$(1_2, 0_1, 3_2, 8_2)$	$(11_2, 5_1, 7_2, 1_2)$	$(8_1, 4_1, 10_2, 3_1)$
$(1_2, 0_2, 9_1, 7_1)$	$(11_2, 6_1, 1_1, 2_2)$	$(8_1, 6_2, 12_1, 4_2)$
$(2_1, 1_1, 8_1, 0_2)$	$(12_1, 9_1, 2_2, 5_2)$	$(9_1, 4_1, 6_2, 11_2)$
$(2_1, 5_1, 0_1, 9_2)$	$(12_1, 9_2, 6_1, 0_2)$	$(9_2, 11_1, 7_1, 4_2)$
$(2_2, 1_2, 4_2, 3_1)$	$(1_1, 3_1, 7_2, 12_1)$	$(0_1, 10_1, 11_1, 6_2)$
$(3_1, 5_1, 0_2, 6_2)$	$(1_2, 10_2, 5_2, 9_2)$	$(10_1, 1_1, 12_2, 4_2)$
$(3_1, 6_1, 7_1, 8_2)$	$(3_1, 0_1, 9_1, 12_2)$	$(10_2, 11_1, 1_1, 8_2)$
$(3_2, 9_2, 1_1, 4_1)$	$(3_1, 5_2, 2_1, 11_1)$	$(10_2, 12_1, 0_1, 7_1)$
$(4_2, 5_1, 9_1, 8_2)$	$(3_2, 9_1, 8_1, 11_1)$	$(11_1, 1_2, 4_1, 12_1)$
$(5_2, 7_1, 1_1, 6_2)$	$(6_1, 2_1, 10_2, 9_1)$	$(12_1, 5_1, 10_1, 3_2)$
$(6_1, 3_2, 4_2, 5_2)$	$(6_2, 1_2, 2_1, 12_2)$	$(12_2, 3_2, 11_2, 7_1)$
$(8_1, 2_2, 7_1, 5_1)$	$(7_1, 2_1, 10_1, 4_1)$	$(12_2, 5_1, 6_1, 11_1)$
$(8_2, 9_2, 2_2, 6_2)$	$(7_2, 5_2, 8_2, 10_1)$	$(4_2, 0_2, 10_2, 11_2)$
$(0_2, 4_1, 12_2, 5_2)$	$(7_2, 6_2, 10_2, 3_2)$	$(8_2, 2_1, 12_1, 11_2)$
$(0_2, 7_2, 11_1, 2_2)$	$(7_2, 8_1, 9_2, 12_2)$	$(9_2, 10_1, 11_2, 3_1)$

For $n = 52$, take a GDC of type 12^4 from Lemma 3.6. Adjoin four ideal points, fill in each of the first three groups together with the ideal points with a GDC of type $1^{12}4^1$ from Lemma 3.8, and fill in the other group together with the ideal points with an optimal $(16, 5, 4)_3$ code to get the desired code. ■

IV. Case of Length $n \equiv 0 \pmod 6$

In this section, we focus our attention on the determination of $A_3(n, 5, 4)$ for $n \equiv 0 \pmod 6$. For $n = 12$, we have the following lower bound.

Lemma 4.1: $A_3(12, 5, 4) \geq U(12, 3) - 1$.

TABLE V
BASE CODEWORDS OF SMALL OPTIMAL $(n, 5, 4)_3$ CODES FOR $n \equiv 0 \pmod 6$

n	Codewords					
18	$(0_1,2_1,5_1,8_1)$	$(1_1,2_1,3_1,9_2)$	$(0_1,4_2,9_2,14_2)$	$(1_1,5_2,8_2,11_2)$	$(0_1,1_2,10_2,11_2)$	$(1_1,10_2,12_2,16_2)$
	$(0_1,9_1,3_2,5_2)$	$(1_1,6_1,3_2,4_2)$	$(1_1,5_1,13_1,0_2)$	$(0_1,17_2,6_2,13_2)$	$(0_1,4_1,11_1,12_2)$	
24	$(0_1,2_2,3_2,5_2)$	$(1_1,3_1,20_2,4_2)$	$(0_1,22_2,7_2,21_2)$	$(0_1,9_2,13_2,16_2)$	$(1_1,2_1,12_2,17_2)$	$(0_1,14_1,15_1,19_1)$
	$(0_1,2_1,6_1,13_1)$	$(1_1,4_1,22_2,5_2)$	$(0_1,23_2,8_2,12_2)$	$(1_1,19_1,8_2,10_2)$	$(1_1,22_1,9_2,15_2)$	$(1_1,15_1,21_2,13_2)$
	$(0_1,8_1,4_2,14_2)$	$(1_1,6_2,19_2,0_2)$	$(0_1,9_1,17_1,19_2)$			
36	$(0_1,2_1,11_1,9_2)$	$(0_1,25_2,2_2,21_2)$	$(0_1,5_2,11_2,14_2)$	$(1_1,30_1,7_2,14_2)$	$(0_1,15_2,26_2,10_2)$	$(1_1,18_1,10_2,13_2)$
	$(1_1,4_1,5_1,14_1)$	$(0_1,31_1,34_2,6_2)$	$(0_1,6_1,22_1,30_2)$	$(1_1,3_1,11_2,31_2)$	$(0_1,23_2,33_2,35_2)$	$(1_1,20_1,25_1,21_2)$
	$(1_1,9_1,23_2,8_2)$	$(0_1,3_1,19_2,27_2)$	$(0_2,9_2,31_2,32_2)$	$(1_1,6_2,16_2,28_2)$	$(0_1,25_1,18_2,29_2)$	$(1_1,20_2,26_2,27_2)$
	$(0_1,21_1,3_2,22_2)$	$(0_1,4_1,12_1,16_2)$	$(1_1,2_1,15_1,31_1)$	$(0_1,15_1,32_2,17_2)$	$(1_1,11_1,32_2,34_2)$	
42	$(0_1,4_2,1_2,26_2)$	$(0_1,5_1,14_1,13_1)$	$(1_1,20_1,14_2,4_1)$	$(0_1,31_2,35_2,30_2)$	$(0_1,41_2,39_2,11_2)$	$(1_1,37_1,27_1,12_1)$
	$(1_1,3_1,6_2,24_2)$	$(0_1,6_2,34_2,38_2)$	$(1_1,28_2,5_2,36_2)$	$(0_1,33_2,17_1,12_2)$	$(1_1,23_2,39_2,33_2)$	$(1_1,37_2,13_2,10_2)$
	$(1_1,3_2,8_2,20_2)$	$(0_1,7_2,20_2,18_2)$	$(1_1,2_2,29_1,22_1)$	$(0_1,35_1,32_2,13_2)$	$(1_1,29_2,14_1,30_2)$	$(1_1,35_2,15_2,32_1)$
	$(0_1,22_1,4_1,27_2)$	$(0_1,8_2,24_2,10_1)$	$(1_1,9_2,21_1,25_1)$	$(0_1,37_1,19_2,28_2)$	$(1_1,32_2,26_2,41_2)$	$(1_1,12_2,13_1,19_2)$
	$(0_1,27_1,37_2,2_2)$	$(1_1,0_1,40_1,34_1)$	$(0_1,21_2,29_2,12_1)$			
48	$(0_1,4_2,3_2,24_2)$	$(0_1,29_2,17_2,6_2)$	$(1_1,28_2,32_1,5_1)$	$(0_1,30_2,21_2,28_2)$	$(0_1,26_1,12_1,19_2)$	$(1_1,47_2,25_2,43_2)$
	$(1_1,3_1,2_1,42_1)$	$(0_1,2_2,15_1,12_2)$	$(1_1,37_2,4_1,41_1)$	$(0_1,31_1,13_1,20_1)$	$(0_1,45_2,13_2,35_1)$	$(1_1,27_1,17_1,42_2)$
	$(1_1,6_2,2_2,21_2)$	$(0_1,4_1,42_1,47_2)$	$(1_1,9_2,22_1,20_1)$	$(0_1,32_1,23_1,22_2)$	$(1_1,35_2,15_2,34_2)$	$(1_1,32_2,43_1,29_1)$
	$(0_1,11_2,26_2,8_2)$	$(1_1,13_2,8_2,46_1)$	$(1_1,20_2,3_2,17_2)$	$(0_1,32_2,46_2,20_2)$	$(0_1,18_1,34_2,43_1)$	$(1_1,37_1,29_2,19_2)$
	$(0_1,1_2,36_2,42_2)$	$(1_1,18_2,5_2,10_2)$	$(0_1,23_2,31_2,25_2)$	$(0_1,33_1,14_2,39_2)$	$(1_1,44_2,12_2,33_2)$	$(1_1,39_2,22_2,30_1)$
	$(0_1,27_2,18_2,5_1)$					
54	$(1_1,24_1,5_2,9_1)$	$(0_1,8_1,44_1,49_2)$	$(1_1,53_2,0_2,23_2)$	$(0_1,25_2,12_2,16_2)$	$(1_1,20_1,50_1,22_1)$	$(1_1,39_1,48_1,41_2)$
	$(1_1,5_1,38_1,6_1)$	$(0_1,9_2,15_2,31_2)$	$(1_1,9_2,12_1,13_2)$	$(0_1,26_2,42_2,23_2)$	$(1_1,24_2,28_1,42_1)$	$(1_1,41_1,50_2,35_1)$
	$(0_1,19_1,48_1,2_2)$	$(1_1,0_1,40_2,34_2)$	$(0_1,11_2,18_2,20_1)$	$(0_1,29_2,28_2,39_2)$	$(1_1,26_2,47_2,44_2)$	$(1_1,45_1,52_2,22_2)$
	$(0_1,20_2,46_2,7_2)$	$(1_1,39_2,37_2,6_2)$	$(0_1,12_1,44_2,50_1)$	$(0_1,30_2,13_2,22_2)$	$(1_1,29_2,43_2,40_1)$	$(1_1,45_2,18_2,20_2)$
	$(0_1,37_2,19_2,3_1)$	$(1_1,43_1,15_2,7_2)$	$(0_1,23_1,51_1,17_2)$	$(0_1,33_2,21_2,14_2)$	$(1_1,31_1,44_1,53_1)$	$(1_1,48_2,28_2,33_2)$
	$(0_1,47_1,4_2,29_1)$	$(1_1,4_2,14_2,46_2)$	$(0_1,24_2,37_1,38_2)$	$(0_1,53_2,10_2,27_2)$	$(1_1,31_2,11_2,36_2)$	
60	$(0_1,4_2,8_2,10_2)$	$(1_1,12_1,29_2,7_2)$	$(0_1,11_2,17_1,41_1)$	$(0_1,43_1,42_2,32_2)$	$(1_1,16_2,51_2,59_2)$	$(1_1,49_1,30_2,33_1)$
	$(0_1,8_1,6_2,51_2)$	$(1_1,16_1,43_1,2_2)$	$(0_1,19_2,15_2,35_2)$	$(0_1,44_2,39_1,26_1)$	$(1_1,18_2,57_2,54_2)$	$(1_1,49_2,40_1,52_2)$
	$(1_1,3_1,39_2,2_1)$	$(1_1,23_2,35_2,8_2)$	$(0_1,27_2,38_2,22_2)$	$(0_1,48_2,26_2,34_2)$	$(1_1,26_1,22_2,47_2)$	$(1_1,54_1,51_1,56_1)$
	$(0_1,23_2,3_1,16_1)$	$(1_1,25_2,11_2,9_1)$	$(0_1,28_2,29_2,48_1)$	$(0_1,49_2,30_2,25_2)$	$(1_1,32_2,53_2,19_2)$	$(1_1,57_1,36_2,24_2)$
	$(0_1,37_1,2_2,15_1)$	$(1_1,41_2,4_2,27_1)$	$(0_1,31_1,38_1,39_2)$	$(0_1,54_2,36_2,47_2)$	$(1_1,33_2,27_2,10_2)$	$(1_1,12_2,46_2,45_2)$
	$(0_1,3_2,13_2,31_2)$	$(1_1,47_1,7_1,38_1)$	$(0_1,33_1,45_2,16_2)$	$(0_1,59_2,50_2,57_2)$	$(1_1,42_1,36_1,56_2)$	$(1_1,43_2,50_1,10_1)$
	$(0_1,52_2,24_2,5_2)$	$(1_1,5_2,34_2,14_2)$	$(0_1,36_1,32_1,50_1)$			
66	$(0_1,7_2,4_1,1_2)$	$(1_1,20_1,43_1,3_2)$	$(1_1,8_2,65_2,33_2)$	$(0_1,30_1,23_2,35_2)$	$(0_1,57_1,30_2,54_2)$	$(1_1,51_1,34_1,29_2)$
	$(0_1,1_1,51_2,3_1)$	$(1_1,30_2,53_2,5_2)$	$(1_1,6_1,20_2,10_2)$	$(0_1,31_1,55_2,20_2)$	$(0_1,27_2,56_2,48_2)$	$(1_1,57_2,17_2,61_2)$
	$(0_1,8_2,2_2,22_2)$	$(1_1,33_1,5_1,59_1)$	$(0_1,15_2,45_2,53_2)$	$(0_1,37_1,13_2,64_2)$	$(0_1,29_2,12_2,19_2)$	$(1_1,60_2,22_2,44_2)$
	$(1_1,54_2,7_2,8_1)$	$(1_1,36_2,24_2,6_2)$	$(0_1,17_2,18_2,50_2)$	$(0_1,39_1,47_2,34_2)$	$(0_1,44_1,40_2,54_1)$	$(1_1,12_2,41_2,14_2)$
	$(0_1,11_2,24_2,9_2)$	$(1_1,39_2,0_2,55_2)$	$(0_1,18_1,38_1,53_1)$	$(0_1,41_2,38_2,21_2)$	$(0_1,51_1,43_2,32_2)$	$(1_1,15_2,24_1,50_2)$
	$(0_1,13_1,2_1,33_2)$	$(1_1,4_1,32_2,38_1)$	$(0_1,25_2,39_2,44_2)$	$(1_1,47_2,38_2,42_2)$	$(1_1,21_1,22_1,31_1)$	$(1_1,35_2,11_2,34_2)$
	$(0_1,6_1,55_1,14_1)$	$(1_1,4_2,45_1,63_2)$	$(0_1,10_2,36_2,19_1)$	$(0_1,42_2,50_1,26_1)$	$(1_1,42_1,13_2,49_1)$	$(1_1,37_2,16_2,15_1)$
	$(0_1,6_2,27_1,21_1)$					
78	$(0_1,2_2,22_2,9_2)$	$(0_1,8_1,58_1,62_1)$	$(0_1,10_1,60_2,76_1)$	$(0_1,51_2,27_1,74_2)$	$(0_1,76_2,68_2,44_2)$	$(1_1,60_1,74_2,54_1)$
	$(1_1,22_2,6_2,8_2)$	$(1_1,13_1,5_1,36_2)$	$(0_1,11_2,18_2,45_2)$	$(0_1,55_2,12_2,16_2)$	$(0_1,77_2,51_1,71_2)$	$(1_1,63_1,23_2,43_1)$
	$(0_1,31_1,6_2,65_2)$	$(1_1,19_1,4_2,30_1)$	$(0_1,13_1,11_1,61_1)$	$(0_1,56_2,47_1,26_2)$	$(1_1,17_2,18_2,12_2)$	$(1_1,64_1,28_2,29_2)$
	$(0_1,36_1,71_1,3_1)$	$(1_1,39_1,9_2,69_2)$	$(0_1,17_2,34_1,66_2)$	$(0_1,59_2,31_2,34_2)$	$(1_1,20_2,65_2,62_2)$	$(1_1,68_2,23_1,22_1)$
	$(0_1,36_2,8_2,25_2)$	$(1_1,3_2,50_2,60_2)$	$(0_1,21_1,19_2,55_1)$	$(0_1,63_2,33_2,10_2)$	$(1_1,34_2,56_2,47_2)$	$(1_1,72_2,55_2,45_2)$
	$(0_1,4_2,64_2,30_2)$	$(1_1,41_2,5_2,67_2)$	$(0_1,35_2,15_2,23_1)$	$(0_1,64_1,39_2,24_2)$	$(1_1,38_1,30_2,33_1)$	$(1_1,75_2,19_2,70_1)$
	$(0_1,73_2,3_2,49_2)$	$(1_1,53_1,44_2,7_2)$	$(0_1,40_2,77_1,60_1)$	$(0_1,67_2,46_1,69_2)$	$(1_1,40_1,16_2,53_2)$	$(1_1,57_2,20_1,73_1)$
	$(0_1,75_2,1_2,65_1)$	$(1_1,55_1,2_2,14_2)$	$(0_1,43_1,38_1,29_1)$	$(0_1,72_2,29_2,41_2)$	$(1_1,51_2,66_2,37_2)$	$(0_1,75_1,48_2,57_2)$
	$(0_1,7_2,47_2,28_2)$	$(1_1,73_2,40_2,0_2)$	$(0_1,48_1,33_1,26_1)$			

TABLE VI
BASE CODEWORDS OF GDC OF TYPE $1^{66}18^1$ IN LEMMA 4.3

Type	Codewords					
$1^{66}18^1$	$(0_1,2_1,79_1,4_2)$	$(0_1,7_1,70_1,53_2)$	$(0_1,10_2,57_2,78_2)$	$(0_1,32_1,45_1,69_1)$	$(1_1,10_1,77_1,44_2)$	$(1_1,38_2,33_2,82_2)$
	$(0_1,6_1,60_2,7_2)$	$(0_1,8_1,68_1,52_2)$	$(0_1,15_2,27_2,61_2)$	$(0_1,35_1,18_2,36_2)$	$(1_1,11_2,24_2,28_2)$	$(1_1,43_2,45_2,67_2)$
	$(1_1,69_1,4_2,5_2)$	$(0_2,77_1,3_2,52_2)$	$(0_1,17_1,46_2,77_2)$	$(0_1,46_1,43_2,79_2)$	$(1_1,12_1,27_1,69_2)$	$(1_1,51_2,59_2,70_2)$
	$(0_1,11_1,6_2,49_2)$	$(1_1,15_1,76_1,3_2)$	$(0_1,19_2,26_2,70_2)$	$(0_1,50_2,42_2,71_2)$	$(1_1,16_2,32_2,79_2)$	$(1_1,53_2,29_2,35_2)$
	$(0_1,1_1,16_1,14_2)$	$(1_1,2_1,22_2,27_2)$	$(0_1,21_1,23_1,39_1)$	$(0_1,52_1,83_1,41_2)$	$(1_1,18_1,80_1,56_2)$	$(1_1,54_2,63_2,83_2)$
	$(0_1,29_1,62_2,8_2)$	$(1_1,4_1,13_1,49_2)$	$(0_1,24_1,36_1,47_2)$	$(0_1,61_1,59_2,73_2)$	$(1_1,19_2,60_2,77_2)$	$(1_1,64_2,26_2,68_2)$
	$(0_1,3_1,17_2,33_2)$	$(1_1,57_2,6_2,74_2)$	$(0_1,25_1,73_1,24_2)$	$(0_1,65_2,13_2,75_2)$	$(1_1,21_1,25_1,31_1)$	$(1_1,66_1,61_2,23_2)$
	$(0_1,48_1,3_2,30_2)$	$(1_1,7_2,10_2,17_2)$	$(0_1,26_1,53_1,83_2)$	$(0_1,66_1,39_2,40_2)$	$(1_1,23_1,67_1,40_2)$	$(1_1,74_1,52_2,21_2)$
	$(0_1,4_1,76_1,35_2)$	$(1_1,81_1,48_2,9_2)$	$(0_1,28_1,47_1,74_2)$	$(0_1,82_1,37_2,58_2)$	$(1_1,24_1,36_2,76_2)$	$(1_2,26_2,37_2,46_2)$
	$(0_1,5_1,74_1,29_2)$	$(1_1,8_1,30_1,66_2)$	$(0_1,28_2,51_2,81_2)$	$(0_2,10_2,36_2,42_2)$	$(1_1,29_1,41_2,80_2)$	$(1_2,69_1,36_2,38_2)$
	$(0_1,5_2,16_2,67_2)$	$(1_1,9_1,79_1,20_2)$	$(0_1,31_1,56_1,22_2)$	$(0_2,22_2,37_2,72_2)$	$(1_1,35_1,82_1,42_2)$	$(1_2,78_1,41_2,45_2)$
	$(0_1,72_1,56_2,9_2)$					

Proof: The desired code is constructed on $\{0,1,2,\ldots,11\}$ with 41 codewords listed next:

$(0_2,1_1,4_1,7_1)$	$(9_1,0_2,5_2,6_1)$	$(3_1,6_1,2_1,10_1)$
$(1_1,5_2,8_1,0_1)$	$(9_1,2_2,3_1,1_1)$	$(3_2,11_2,4_1,2_2)$
$(1_2,2_1,4_1,8_1)$	$(9_2,0_2,1_2,3_2)$	$(5_1,0_2,3_1,10_2)$
$(1_2,2_2,0_1,6_2)$	$(0_1,10_2,9_1,2_1)$	$(5_2,9_2,10_1,2_2)$
$(3_1,8_1,9_2,7_1)$	$(0_1,11_2,3_1,8_2)$	$(6_2,10_2,4_2,8_1)$
$(3_2,6_1,4_2,7_2)$	$(0_1,6_1,4_1,11_1)$	$(6_2,1_1,7_2,10_1)$
$(4_1,6_2,3_1,5_2)$	$(10_1,5_1,4_1,8_2)$	$(7_1,0_1,10_1,3_2)$
$(4_2,2_1,7_1,5_2)$	$(11_1,2_1,5_1,1_1)$	$(7_2,9_2,10_2,4_1)$
$(4_2,8_2,0_2,2_2)$	$(11_1,3_1,1_2,4_2)$	$(9_1,11_1,6_2,3_2)$
$(5_1,9_1,7_2,1_2)$	$(11_1,5_2,7_2,8_2)$	$(10_1,8_1,11_1,0_2)$
$(5_1,9_2,4_2,0_1)$	$(11_2,0_2,7_2,2_1)$	$(4_2,9_1,11_2,10_1)$
$(6_1,7_1,1_2,8_2)$	$(11_2,1_1,6_1,9_2)$	$(5_2,10_2,11_2,1_2)$
$(6_2,9_2,2_1,8_2)$	$(11_2,6_2,5_1,7_1)$	$(7_1,10_2,2_2,11_1)$
$(8_1,2_2,6_1,5_1)$	$(1_1,3_2,8_2,10_2)$.	

■

Lemma 4.2: $A_3(n,5,4) = U(n,3)$ for each $n \equiv 0 \pmod 6$, $18 \le n \le 66$, or $n = 78$.

Proof: For $n = 30$, the required code is constructed in Example 2.7. For the other lengths n, the required codes are constructed on $\mathbb{Z}_n$, and obtained by developing the elements of $\mathbb{Z}_n$ in the codewords listed in Table V $+2 \pmod n$. ■

Lemma 4.3: There exists a GDC of type $1^{66}18^1$ and size 2211.

Proof: Let $X = \{0,1,\ldots,83\}$, and $\mathcal{G} = \{\{i\} : 0 \le i \le 65\} \cup \{\{66,67,\ldots,83\}\}$. Then $(X,\mathcal{G},\mathcal{C})$ is a GDC of type $1^{66}18^1$ and size 2211, where $\mathcal{C}$ is generated from the codewords listed in Table VI, which are developed under the automorphism group $\langle(0\ 2\ 4\ \cdots\ 64)(1\ 3\ 5\ \cdots\ 65)(66\ 67\ 68)(69\ 70\ 71)(72\ 73\ 74)(75\ 76\ 77)(78\ 79\ 80)(81\ 82\ 83)\rangle$. ■

Lemma 4.4: There exists a GDC of type 18^u and size $108u(u-1)$ for all $u \ge 4$.

Proof: For $u \ge 4$ and $u \ne 6$, the proof is similar to that of Lemma 3.6. Here, the input GDCs of types 6^4 and 6^7 are from Lemmas 2.6 and 3.3.

For $u = 6$, inflate a GDC of type 6^6 (see Lemma 3.3) by 3 to obtain the required GDC. ■

Corollary 4.1: $A_3(18u,5,4) = U(18u,3)$ for all $u \ge 4$.

Proof: Take GDCs of type 18^u from Lemma 4.4 and fill in all the groups with optimal $(18,5,4)_3$ codes from Lemma 4.2. The results are optimal $(18u,5,4)_3$ codes for all $u \ge 4$. ■

Lemma 4.5: $A_3(18u+30,5,4) = U(18u+30,3)$ for each $u \in \{3,4\}$.

Proof: For $u = 3$, take the GDC of type $1^{66}18^1$ in Lemma 4.3 and fill in the group of size 18 with an optimal $(18,5,4)_3$ code to get the desired code.

For $u = 4$, the desired code is constructed in Example 2.7. ■

Lemma 4.6: There exist GDCs of type $18^u m^1$ and size $108u(u-1)+12um$ for all $u \ge 31$ with $m \in \{24,30\}$.

Proof: Take a $\mathrm{TD}(6,3t)$ from Lemma 2.4, and apply the Fundamental Construction with weight 6 to all points in the first four groups, $3x$ points in the fifth group, and y points in the last group, where $x = 0$ or $3 \le x \le t$, $y \in \{1,2\}$. The other points are given weight 0. The input GDCs of types 6^4, 6^5, and 6^6 are from Lemmas 2.6 and 3.3. The result is a GDC of type $(18t)^4(18x)^1(6y)^1$. Adjoin 18 ideal points, and fill in the first five groups together with the ideal points with GDCs of types 18^{t+1} or 18^{x+1} to get a GDC of type $18^{4t+x}(6y+18)^1$. The result is a GDC of type $18^u m^1$ with $m \in \{24,30\}$, where $u = 4t+x$ can take any integer no less than 31. ■

Corollary 4.2: $A_3(18u+m,5,4) = U(18u+m,3)$ for all $u \ge 31$ with $m \in \{24,30\}$.

Proof: Take the GDCs of types $18^u m^1$ for all $u \ge 31$ with $m \in \{24,30\}$ from Lemma 4.6. Fill in the groups of the GDCs with suitable codes of small lengths from Lemma 4.2 to obtain the desired codes. ■

Lemma 4.7: There exist GDCs of type $18^u m^1$ and size $108u(u-1)+12um$ for each $u \equiv 0 \pmod 3$, $12 \le u \le 27$, with $m \in \{24,30,42,48,60,66\}$.

Proof: Take a $\mathrm{TD}(10,9)$ from Lemma 2.4 and apply the Fundamental Construction with weight 6 to all points in the first four groups, y points in the last group, and $3x_i$ points in the ith remaining group for $1 \le i \le 5$, and weight 0 to all the other points to get a GDC of type $54^4(18x_1)^1\cdots(18x_5)^1(6y)^1$ for $x_i \in \{0,3\}$, $y \in \{1,2,4,5,7,8\}$. The input GDCs of type 6^s for $s \in [4,10]$ are from Lemmas 2.6 and 3.3. Adjoin 18 ideal points, and fill in the groups together with the ideal points with GDCs of type 18^4 except for the group of size $6y$ to get a GDC of type $18^{12+\sum x_i}(6y+18)^1$ for $x_i \in \{0,3\}$, $y \in \{1,2,4,5,7,8\}$. The result is a GDC of type $18^u m^1$ for $u \equiv 0 \pmod 3$, $12 \le u \le 27$, with $m \in \{24,30,42,48,60,66\}$. ■

Corollary 4.3: $A_3(18u+m,5,4) = U(18u+m,3)$ for each $12 \le u \le 29$ with $m \in \{24,30\}$.

Proof: Take the GDCs of types $18^u m^1$ for each $u \equiv 0 \pmod 3$, $12 \le u \le 27$, with $m \in \{24,30,42,48,60,66\}$ from Lemma 4.7. Fill in the groups of the GDCs with suitable codes of small lengths from Lemma 4.2 to obtain the desired codes. ■

Lemma 4.8: The following GDCs all exist:

1) type 24^u and size $192u(u-1)$ for $u \in \{4,5,7,8\}$;
2) type 30^u and size $300u(u-1)$ for $u \in \{5,7,19\}$;
3) type 24^u36^1 and size $192u(u+2)$ for $u \in \{4,5,7,8,22\}$;
4) type 24^u18^1 and size $96u(2u+1)$ for $u \in \{4,5,7\}$;
5) type 24^830^1 and size 14592;
6) type 30^524^1 and size 8400.

Proof: The required GDCs of types 24^u for $u \in \{4,5,7,8\}$ are obtained by inflating GDCs of types 6^u (see Lemmas 2.6 and 3.3) by 4.

The GDCs of types 30^u for $u \in \{5,7,19\}$ are obtained by inflating GDCs of types 6^u (see Lemmas 2.6 and 3.3) by 5.

The GDCs of types 24^u36^1 for $u \in \{4,5,7,8,22\}$ are obtained by applying the Fundamental Construction with weight 4 to $\{4\}$-GDDs of type 6^u9^1 (see [20, Th. 1.6]).

For GDCs of types 24^u18^1 for $u \in \{4,5,7\}$, take a $\mathrm{TD}(5,u)$ from Lemma 2.4, and remove a point to get a $\{5,u\}$-GDD of type $4^u(u-1)^1$. Apply the Fundamental Construction with

TABLE VII
BASE CODEWORDS OF SMALL OPTIMAL $(n, 5, 4)_3$ CODES FOR $n \equiv 5 \pmod 6$

n	Codewords
17	$(0_1, 1_1, 3_1, 7_1)$ $(0_1, 5_1, 1_2, 2_2)$ $(0_1, 3_2, 5_2, 10_2)$ $(0_1, 6_2, 9_2, 15_2)$ $(0_1, 8_1, 12_2, 16_2)$
23	$(0_1, 1_1, 3_1, 7_1)$ $(0_1, 1_2, 2_2, 17_2)$ $(0_1, 5_2, 7_2, 19_2)$ $(0_1, 6_2, 10_2, 16_2)$ $(0_1, 9_1, 18_2, 21_2)$ $(0_1, 10_1, 15_1, 14_2)$ $(0_1, 11_1, 3_2, 8_2)$
29	$(0_1, 10_1, 7_2, 8_2)$ $(0_1, 22_1, 6_2, 16_2)$ $(0_1, 2_1, 14_1, 25_1)$ $(0_1, 9_2, 14_2, 18_2)$ $(0_1, 20_1, 15_2, 22_2)$ $(0_1, 10_2, 12_2, 28_2)$ $(0_1, 5_1, 13_1, 1_2)$ $(0_1, 28_1, 3_2, 20_2)$ $(0_1, 5_2, 11_2, 19_2)$
35	$(0_1, 1_1, 3_1, 34_2)$ $(0_1, 9_1, 20_1, 1_2)$ $(0_1, 2_2, 12_2, 14_2)$ $(0_1, 5_1, 13_1, 23_1)$ $(0_1, 8_2, 15_2, 19_2)$ $(0_1, 21_2, 26_2, 29_2)$ $(0_1, 4_1, 9_2, 22_2)$ $(0_1, 19_1, 4_2, 30_2)$ $(0_1, 3_2, 17_2, 23_2)$ $(0_1, 7_1, 13_2, 32_2)$ $(0_1, 21_1, 10_2, 28_2)$
41	$(0_1, 1_1, 9_2, 23_2)$ $(0_1, 18_1, 7_2, 10_2)$ $(0_1, 5_1, 13_1, 24_1)$ $(0_1, 3_1, 35_1, 32_2)$ $(0_1, 34_1, 20_2, 28_2)$ $(0_1, 25_2, 34_2, 36_2)$ $(0_1, 4_2, 5_2, 26_2)$ $(0_1, 39_1, 1_2, 19_2)$ $(0_1, 6_2, 13_2, 18_2)$ $(0_1, 21_1, 25_1, 37_2)$ $(0_1, 11_2, 15_2, 39_2)$ $(0_1, 14_2, 24_2, 40_2)$ $(0_1, 12_1, 26_1, 2_2)$
47	$(0_1, 3_2, 9_2, 45_2)$ $(0_1, 2_1, 21_1, 43_1)$ $(0_1, 7_2, 32_2, 34_2)$ $(0_1, 9_1, 16_1, 22_2)$ $(0_1, 14_2, 30_2, 37_2)$ $(0_1, 14_1, 26_2, 29_2)$ $(0_1, 10_1, 5_2, 33_2)$ $(0_1, 4_2, 18_2, 19_2)$ $(0_1, 8_1, 32_1, 35_1)$ $(0_1, 11_1, 21_2, 38_2)$ $(0_1, 17_1, 28_2, 41_2)$ $(0_1, 17_2, 25_2, 35_2)$ $(0_1, 1_1, 40_2, 44_2)$ $(0_1, 5_1, 34_1, 36_2)$ $(0_1, 8_2, 20_2, 46_2)$
53	$(0_1, 9_1, 1_2, 4_2)$ $(0_1, 1_1, 24_1, 51_1)$ $(0_1, 11_1, 21_1, 36_1)$ $(0_1, 13_1, 46_1, 31_2)$ $(0_1, 31_1, 49_1, 17_2)$ $(0_1, 41_1, 23_2, 46_2)$ $(0_1, 5_1, 7_2, 52_2)$ $(0_1, 8_2, 30_2, 32_2)$ $(0_1, 11_2, 22_2, 27_2)$ $(0_1, 15_2, 16_2, 36_2)$ $(0_1, 37_1, 24_2, 43_2)$ $(0_1, 45_1, 33_2, 42_2)$ $(0_1, 6_1, 9_2, 19_2)$ $(0_1, 10_2, 37_2, 49_2)$ $(0_1, 12_2, 25_2, 29_2)$ $(0_1, 26_2, 44_2, 51_2)$ $(0_1, 39_1, 14_2, 20_2)$
59	$(0_1, 2_2, 23_2, 53_2)$ $(0_1, 8_1, 43_1, 14_2)$ $(0_1, 15_1, 57_1, 50_2)$ $(0_1, 25_1, 53_1, 58_1)$ $(0_1, 45_1, 55_1, 12_2)$ $(0_1, 13_2, 38_2, 40_2)$ $(0_1, 3_2, 42_2, 57_2)$ $(0_1, 9_1, 10_2, 56_2)$ $(0_1, 15_2, 21_2, 37_2)$ $(0_1, 29_2, 41_2, 48_2)$ $(0_1, 46_1, 18_2, 36_2)$ $(0_1, 22_1, 40_1, 27_2)$ $(0_1, 4_2, 28_2, 32_2)$ $(0_1, 11_1, 19_2, 20_2)$ $(0_1, 20_1, 27_1, 56_1)$ $(0_1, 38_1, 24_2, 33_2)$ $(0_1, 47_1, 22_2, 39_2)$ $(0_1, 44_2, 55_2, 58_2)$ $(0_1, 7_2, 17_2, 43_2)$
65	$(0_1, 11_1, 1_2, 5_2)$ $(0_1, 4_1, 33_1, 58_2)$ $(0_1, 13_2, 22_2, 60_2)$ $(0_1, 21_1, 35_2, 61_2)$ $(0_1, 42_1, 62_1, 30_2)$ $(0_1, 49_1, 10_2, 64_2)$ $(0_1, 24_1, 3_2, 6_2)$ $(0_1, 4_2, 18_2, 28_2)$ $(0_1, 16_2, 37_2, 56_2)$ $(0_1, 23_2, 36_2, 52_2)$ $(0_1, 43_1, 48_1, 12_2)$ $(0_1, 41_2, 46_2, 48_2)$ $(0_1, 2_1, 9_1, 59_1)$ $(0_1, 55_1, 9_2, 21_2)$ $(0_1, 17_2, 45_2, 51_2)$ $(0_1, 26_1, 40_1, 53_1)$ $(0_1, 46_1, 24_2, 57_2)$ $(0_1, 47_1, 32_2, 49_2)$ $(0_1, 1_1, 31_1, 39_2)$ $(0_1, 7_2, 27_2, 42_2)$ $(0_1, 20_2, 62_2, 63_2)$
71	$(0_1, 2_1, 3_1, 5_2)$ $(0_1, 7_1, 33_1, 25_2)$ $(0_1, 15_1, 56_2, 67_2)$ $(0_1, 20_1, 32_2, 33_2)$ $(0_1, 28_2, 31_2, 57_2)$ $(0_1, 40_2, 64_2, 70_2)$ $(0_1, 4_2, 9_2, 65_2)$ $(0_1, 8_2, 20_2, 47_2)$ $(0_1, 15_2, 22_2, 53_2)$ $(0_1, 21_2, 42_2, 46_2)$ $(0_1, 29_2, 48_2, 66_2)$ $(0_1, 50_1, 61_1, 17_2)$ $(0_1, 14_1, 1_2, 44_2)$ $(0_1, 10_2, 24_2, 59_2)$ $(0_1, 16_1, 65_1, 55_2)$ $(0_1, 24_1, 37_1, 60_2)$ $(0_1, 32_1, 11_2, 69_2)$ $(0_1, 62_1, 45_2, 68_2)$ $(0_1, 43_1, 7_2, 62_2)$ $(0_1, 12_1, 41_1, 66_1)$ $(0_1, 19_1, 27_1, 67_1)$ $(0_1, 26_2, 34_2, 43_2)$ $(0_1, 36_1, 14_2, 16_2)$
83	$(0_1, 2_1, 72_1, 31_2)$ $(0_1, 4_2, 33_2, 67_2)$ $(0_1, 8_2, 16_2, 47_2)$ $(0_1, 22_1, 62_2, 71_2)$ $(0_1, 36_2, 48_2, 58_2)$ $(0_1, 65_2, 66_2, 81_2)$ $(0_1, 3_1, 19_1, 79_2)$ $(0_1, 5_1, 15_2, 39_2)$ $(0_1, 12_2, 50_2, 82_2)$ $(0_1, 22_2, 24_2, 43_2)$ $(0_1, 37_1, 73_1, 78_2)$ $(0_1, 75_1, 27_2, 64_2)$ $(0_1, 3_2, 30_2, 56_2)$ $(0_1, 60_1, 2_2, 74_2)$ $(0_1, 14_1, 56_1, 75_2)$ $(0_1, 26_1, 11_2, 80_2)$ $(0_1, 45_1, 54_1, 63_2)$ $(0_1, 34_1, 65_1, 82_1)$ $(0_1, 40_1, 1_2, 77_2)$ $(0_1, 6_1, 50_1, 26_2)$ $(0_1, 15_1, 21_2, 38_2)$ $(0_1, 30_1, 51_1, 58_1)$ $(0_1, 45_2, 70_2, 73_2)$ $(0_1, 46_2, 51_2, 69_2)$ $(0_1, 4_1, 63_1, 32_2)$ $(0_1, 7_2, 13_2, 55_2)$ $(0_1, 17_2, 53_2, 57_2)$
89	$(0_1, 5_2, 7_2, 14_2)$ $(0_1, 83_1, 6_2, 36_2)$ $(0_1, 18_2, 44_2, 64_2)$ $(0_1, 32_1, 71_1, 11_2)$ $(0_1, 48_2, 71_2, 77_2)$ $(0_1, 65_2, 75_2, 79_2)$ $(0_1, 1_2, 59_2, 87_2)$ $(0_1, 9_1, 14_1, 66_2)$ $(0_1, 20_2, 62_2, 84_2)$ $(0_1, 35_2, 40_2, 88_2)$ $(0_1, 49_1, 62_1, 43_2)$ $(0_1, 66_1, 15_2, 30_2)$ $(0_1, 2_1, 10_1, 70_1)$ $(0_1, 10_2, 49_2, 61_2)$ $(0_1, 26_1, 85_1, 72_2)$ $(0_1, 41_1, 21_2, 86_2)$ $(0_1, 51_1, 88_1, 22_2)$ $(0_1, 69_1, 13_2, 34_2)$ $(0_1, 2_2, 39_2, 74_2)$ $(0_1, 15_1, 24_2, 32_2)$ $(0_1, 31_1, 43_1, 67_1)$ $(0_1, 44_1, 78_1, 47_2)$ $(0_1, 51_2, 67_2, 85_2)$ $(0_1, 86_1, 16_2, 78_2)$ $(0_1, 4_2, 37_2, 82_2)$ $(0_1, 16_1, 33_1, 41_2)$ $(0_1, 31_2, 50_2, 63_2)$ $(0_1, 47_1, 82_1, 73_2)$ $(0_1, 61_1, 27_2, 28_2)$

weight 6 to all the points in the groups of size 4 and three points in the group of size $u-1$, and weight 0 to the remaining points. The result is a GDC of type $24^u 18^1$.

For a GDC of type $24^8 30^1$, take a $\mathrm{TD}(5,8)$ and remove a point to get a $\{5,8\}$-GDD of type $4^8 7^1$. Apply the Fundamental Construction with weight 6 to all the points in the groups of size 4 and five points in the group of size 7, and weight 0 to the remaining points to obtain a GDC of type $24^8 30^1$.

For a GDC of type $30^5 24^1$, take a $\mathrm{TD}(6,5)$, apply the Fundamental Construction with weight 6 to all the points in the first five groups, 4 points in the last group, and weight 0 to the remaining points to obtain a GDC of type $30^5 24^1$. ■

Corollary 4.4: $A_3(18u+m, 5, 4) = U(18u+m, 3)$ for each $u \in [5, 11] \cup \{30\}$ with $m \in \{24, 30\}$ or $(u, m) = (4, 24)$.

Proof: Take the GDCs constructed in Lemma 4.8. Fill in the groups of the GDCs with suitable codes of small lengths from Lemma 4.2 to obtain the desired codes. ■

Summarizing Lemmas 4.1, 4.2, and 4.5, and Corollaries 4.1 – 4.4, we obtain the main result of this section.

Theorem 4.1: $A_3(n, 5, 4) = U(n, 3)$ for all integer $n \equiv 0 \pmod 6$, $n \geq 18$; $A_3(12, 5, 4) \geq U(12, 3) - 1$.

V. CASE OF LENGTH $n \equiv 2 \pmod 6$

In this section, we will determine the value of $A_3(n, 5, 4)$ for all $n \equiv 2 \pmod 6$. It is easy to prove that if there exists a GDC of type 2^u and size $\frac{4u(u-1)}{3}$ for $u \equiv 1 \pmod 3$, then we have an optimal $(2u, 5, 4)_3$ code.

Lemma 5.1: There exists a GDC of type 2^u and size $\frac{4u(u-1)}{3}$ for each $u \equiv 1 \pmod 3$, $7 \leq u \leq 34$, or $u \in \{40, 43, 52\}$.

Proof: For $u \in \{7, 10, 13, 16, 19, 22, 28, 40\}$, the required GDCs are from Examples 2.1 and 2.5, and Lemmas 2.6 and 3.1.

For each $u \in \{25, 31, 43\}$, take a GDC of type 12^s for $s \in \{4, 5, 7\}$ (see Lemma 3.6). Adjoin two ideal points and fill in the groups together with the ideal points with GDCs of type 2^7 to obtain a GDC of type 2^u.

For each $u \in \{34, 52\}$, take a GDC of type $12^s 18^1$ for $s \in \{4, 7\}$ (see Lemma 3.7), adjoin two ideal points, and fill in the groups together with the ideal points with GDCs of types 2^7 or 2^{10} to obtain the required GDC. ■

Lemma 5.2: There exists a GDC of type 2^u and size $\frac{4u(u-1)}{3}$ for each $u \equiv 1 \pmod 3$, $u \in \{37, 46, 49\}$ or $u \geq 55$.

Proof: Take the GDCs of types $g^t m^1$ for $g \in \{18, 24, 30\}$ and $m \in \{18, 24, 30, 42, 48, 60, 66\}$ constructed in Lemmas

TABLE VIII
BASE CODEWORDS OF GDCS IN LEMMA 6.3

Types	Codewords					
$2^9 5^1$	$(1_1,4_1,2_2,8_2)$	$(1_1,2_1,22_1,6_2)$	$(0_1,3_2,11_2,19_2)$	$(1_1,8_1,19_1,15_2)$	$(2_1,4_1,10_2,19_2)$	$(1_1,11_1,14_2,18_2)$
	$(1_2,6_2,7_2,9_2)$	$(2_1,5_1,6_1,12_1)$	$(0_1,5_1,15_2,21_2)$	$(1_1,9_1,18_1,13_2)$	$(2_2,3_2,15_2,17_2)$	$(1_1,12_1,13_1,15_1)$
	$(2_1,8_1,4_2,7_2)$	$(0_1,13_2,6_2,18_2)$	$(1_1,14_1,9_2,22_2)$	$(2_1,16_1,21_1,0_2)$	$(0_1,13_1,12_2,20_2)$	$(1_1,20_1,12_2,16_2)$
	$(0_1,19_1,5_2,7_2)$	$(0_1,21_1,14_2,1_2)$	$(1_1,4_2,11_2,21_2)$	$(2_1,18_1,14_2,3_2)$	$(0_1,16_2,17_2,22_2)$	$(2_1,13_2,17_2,20_2)$
	$(0_1,2_1,20_1,8_2)$	$(0_1,22_1,10_2,2_2)$				
$2^{12} 5^1$	$(0_1,3_1,9_1,8_2)$	$(1_1,4_2,9_2,27_2)$	$(1_1,15_1,8_2,18_2)$	$(2_1,26_1,19_2,3_2)$	$(2_2,24_1,10_2,0_2)$	$(1_1,21_1,15_2,28_2)$
	$(1_1,0_2,5_2,7_2)$	$(1_1,6_1,28_1,3_2)$	$(1_1,4_1,10_1,20_1)$	$(2_1,28_1,8_2,13_2)$	$(2_2,5_2,16_2,20_2)$	$(1_1,21_2,17_2,25_2)$
	$(0_1,22_2,4_2,7_2)$	$(2_1,5_1,6_1,11_1)$	$(1_1,9_1,20_2,26_2)$	$(2_1,4_1,27_1,17_2)$	$(0_1,22_1,25_1,13_2)$	$(2_1,19_1,16_2,24_2)$
	$(0_1,2_1,1_2,25_2)$	$(0_1,20_2,9_2,24_2)$	$(2_1,12_1,13_1,7_2)$	$(2_1,9_1,11_2,27_2)$	$(0_1,27_1,15_2,16_2)$	$(2_1,21_2,10_2,26_2)$
	$(0_1,8_1,6_2,10_2)$	$(0_1,7_1,11_1,24_1)$	$(2_1,22_2,5_2,28_2)$	$(2_1,9_2,15_2,18_2)$	$(1_1,10_2,11_2,12_2)$	$(2_1,25_1,12_2,20_2)$
	$(1_1,2_1,23_2,6_2)$	$(1_1,12_1,26_1,2_2)$				
$2^{15} 5^1$	$(0_1,4_1,6_1,2_2)$	$(2_1,26_2,5_2,8_2)$	$(0_1,7_2,18_2,33_2)$	$(1_1,32_1,26_2,3_2)$	$(2_1,15_1,32_1,1_2)$	$(0_1,10_2,23_2,31_2)$
	$(2_2,6_2,7_2,8_2)$	$(2_1,7_1,3_2,21_2)$	$(0_2,7_2,10_2,17_2)$	$(1_1,4_1,10_1,17_2)$	$(2_1,31_1,4_2,20_2)$	$(0_1,12_1,14_1,11_2)$
	$(0_1,1_2,9_2,13_2)$	$(0_1,16_1,28_2,4_2)$	$(1_1,14_1,5_2,33_2)$	$(1_1,5_1,18_2,31_2)$	$(2_1,33_1,12_2,7_2)$	$(0_1,20_2,22_2,34_2)$
	$(0_1,3_1,13_1,6_2)$	$(0_1,19_1,33_1,8_2)$	$(1_1,2_1,34_1,28_2)$	$(1_1,6_1,25_2,32_2)$	$(2_1,5_1,22_2,13_2)$	$(0_1,23_1,27_2,30_2)$
	$(0_1,8_1,9_1,20_1)$	$(0_1,1_1,31_1,21_2)$	$(1_1,2_2,10_2,12_2)$	$(1_1,7_2,11_2,30_2)$	$(2_1,6_1,13_1,18_2)$	$(0_2,21_2,24_2,29_2)$
	$(1_1,8_1,0_2,34_2)$	$(0_1,34_1,24_2,5_2)$	$(1_1,30_1,9_2,22_2)$	$(2_1,11_1,9_2,25_2)$	$(2_1,8_1,10_1,22_1)$	$(2_1,27_2,11_2,32_2)$
	$(1_1,9_1,4_2,23_2)$	$(0_1,5_1,30_1,17_2)$				

TABLE IX
BASE CODEWORDS OF GDC OF TYPE $1^{42}15^1$ IN LEMMA 7.2

Types	Codewords					
$1^{42}15^1$	$(0_1,17_2,6_2,8_2)$	$(0_1,4_1,43_1,22_2)$	$(1_1,31_2,2_2,43_2)$	$(1_1,9_1,16_2,55_2)$	$(0_1,20_2,13_2,55_2)$	$(1_1,11_1,20_1,47_1)$
	$(0_1,48_1,2_2,5_2)$	$(0_1,51_1,23_2,7_2)$	$(1_1,34_2,7_2,52_2)$	$(1_2,2_2,15_2,48_2)$	$(0_1,24_1,30_1,40_1)$	$(1_1,14_2,24_2,44_2)$
	$(0_1,5_1,1_2,49_2)$	$(0_1,52_1,4_2,12_2)$	$(1_1,3_1,17_1,25_2)$	$(0_1,11_1,32_2,21_2)$	$(0_1,29_1,54_1,19_2)$	$(1_1,38_2,19_2,53_2)$
	$(0_1,9_2,33_2,3_2)$	$(0_1,8_1,35_2,43_2)$	$(1_1,49_1,35_2,3_2)$	$(0_1,13_1,17_1,53_1)$	$(0_1,31_1,16_2,46_2)$	$(1_1,43_1,41_2,21_2)$
	$(1_1,2_1,0_2,45_2)$	$(0_1,9_1,49_1,14_2)$	$(1_1,4_1,46_1,29_2)$	$(0_1,14_1,11_2,53_2)$	$(0_1,45_1,37_2,41_2)$	$(1_1,44_1,15_2,40_2)$
	$(1_2,4_2,9_2,54_2)$	$(0_2,45_1,4_2,26_2)$	$(1_1,6_1,50_1,32_2)$	$(0_1,15_1,34_2,50_2)$	$(0_1,55_1,24_2,38_2)$	$(1_1,52_1,17_2,26_2)$
	$(0_1,1_1,42_1,10_2)$	$(1_1,18_1,4_2,48_2)$	$(1_1,7_1,19_1,36_2)$	$(0_1,19_1,30_2,51_2)$	$(0_1,56_1,31_2,36_2)$	$(1_2,16_2,41_2,45_2)$
	$(0_1,3_1,15_2,45_2)$	$(1_1,21_1,54_1,5_2)$	$(1_1,8_1,37_2,42_2)$	$(0_1,20_1,27_1,54_2)$	$(0_2,12_2,18_2,19_2)$	

4.4, 4.6 – 4.8, adjoin two ideal points, and fill in the groups of size g or m together with the ideal points with GDCs of types $2^{\frac{g}{2}+1}$ or $2^{\frac{m}{2}+1}$ which are from Lemma 5.1. The results are GDCs of type $2^{\frac{gt+m}{2}+1}$. For each desired $u = \frac{gt+m}{2} + 1$, we list the type and source of the starting GDC in Table II. ■

Combining Lemmas 5.1 and 5.2, we have the following result.

Theorem 5.1: $A_3(n,5,4) = U(n,3)$ for all integer $n \equiv 2 \pmod 6$, $n \geq 14$.

VI. CASE OF LENGTH $n \equiv 5 \pmod 6$

In this section, we determine the value of $A_3(n,5,4)$ for all $n \equiv 5 \pmod 6$.

Lemma 6.1: $A_3(11,5,4) = U(11,3)$.

Proof: The desired code is constructed on $\{0,1,2,\ldots,10\}$ with 33 codewords listed next:

$$\begin{array}{lll}
(0_1,8_1,2_2,7_1) & (3_2,8_1,9_2,7_2) & (0_2,2_1,10_1,1_1) \\
(0_2,5_1,8_1,1_2) & (5_1,6_2,8_2,7_1) & (10_1,4_1,6_2,0_1) \\
(1_1,3_2,5_1,9_1) & (6_1,0_1,9_1,1_2) & (10_1,5_1,3_1,2_2) \\
(1_1,7_2,6_1,4_1) & (6_1,2_2,4_2,9_2) & (1_1,7_1,9_2,10_2) \\
(1_1,8_2,5_2,2_2) & (6_2,3_1,0_2,9_2) & (5_1,7_2,10_2,4_2) \\
(2_1,4_2,7_1,1_2) & (6_2,3_2,5_2,4_2) & (5_2,10_2,0_2,6_1) \\
(2_1,6_2,9_1,7_2) & (8_1,3_1,4_2,1_1) & (6_1,7_1,10_1,3_2) \\
(2_2,0_2,3_2,4_1) & (9_1,4_2,8_2,0_2) & (6_2,1_2,2_2,10_2) \\
(3_1,4_1,9_1,7_1) & (9_2,0_1,5_1,2_1) & (7_2,1_2,10_1,8_2) \\
(3_1,5_2,0_1,7_2) & (9_2,1_2,4_1,5_2) & (8_1,2_1,10_2,4_1) \\
(3_1,6_1,8_2,2_1) & (0_1,10_2,8_2,3_2) & (9_1,5_2,8_1,10_1).
\end{array}$$

■

Lemma 6.2: $A_3(n,5,4) = U(n,3)$ for each $n \equiv 5 \pmod 6$, $17 \leq n \leq 71$, or $n \in \{83, 89\}$.

Proof: The required codes are constructed on $\mathbb{Z}_n$, and obtained by developing the elements of $\mathbb{Z}_n$ in the codewords listed in Table VII $+1 \pmod n$. ■

Lemma 6.3: There exists a GDC of type $2^u 5^1$ and size $\frac{4u(u+4)}{3}$ for each $u \in \{9,12,15\}$.

Proof: For each $u \in \{9,12,15\}$, let $X_u = \{0,1,\ldots,2u+4\}$, and $\mathcal{G}_u = \{\{i, i+u\} : 0 \leq i \leq u-1\} \cup \{\{2u, 2u+1, \ldots, 2u+4\}\}$. Then, $(X_u, \mathcal{G}_u, \mathcal{C}_u)$ is a GDC of type $2^u 5^1$ and size $\frac{4u(u+4)}{3}$, where $\mathcal{C}$ is generated from the codewords listed in Table VIII, which are developed under the automorphism group $\langle (0\ 3\ 6\ \cdots\ 2u-3)(1\ 4\ 7\ \cdots\ 2u-2)(2\ 5\ 8\ \cdots\ 2u-1)(2u)(2u+1)(2u+2)(2u+3)(2u+4)\rangle$. ■

Lemma 6.4: $A_3(6u+5,5,4) = U(6u+5,3)$ for $u = 12$ and all $u \geq 15$.

Proof: For $u = 17$, take a TD$(5,4)$ from Lemma 2.4, and apply the Fundamental Construction by giving weight 6 to all the points in the first four groups and one point in the last group and weight 0 to the remaining points to get a GDC of type $24^4 6^1$. Adjoin five ideal points, fill in the groups of size 24 together with the extra points with a GDC of type $2^{12}5^1$, and fill in the group of size 6 together with the extra points with an optimal code of length 11 to get the desired code.

For $u \in \{12, 15, 16\}$ or $u \geq 18$, the proof is similar to that of Lemma 5.2. Take the GDCs of types $g^t m^1$ for $g \in \{18, 24, 30\}$ and $m \in \{18, 24, 30, 42, 48, 60, 66\}$ constructed in Lemmas

TABLE X
BASE CODEWORDS OF SOME SMALL GDCs IN LEMMA 7.3

Types	Codewords					
$1^{12}3^1$	$(0_1,1_1,2_1,5_2)$	$(0_1,7_1,12_1,4_2)$	$(1_1,5_1,13_1,7_2)$	$(2_1,6_2,11_2,1_2)$	$(1_1,10_1,6_2,12_2)$	$(3_1,4_1,10_1,13_2)$
	$(3_1,5_1,1_2,4_2)$	$(0_2,1_2,8_2,12_2)$	$(2_1,11_1,4_2,7_2)$	$(3_1,5_2,9_2,12_2)$	$(1_1,3_1,10_2,14_2)$	$(3_2,4_2,11_2,12_2)$
	$(0_1,4_1,14_1,1_2)$	$(1_1,14_1,2_2,8_2)$	$(2_1,14_1,3_2,9_2)$	$(3_2,13_1,1_2,2_2)$	$(3_1,11_1,12_1,2_2)$	$(0_1,11_2,2_2,13_2)$
	$(0_1,6_2,8_2,10_2)$	$(1_1,4_1,7_1,11_2)$	$(2_1,6_1,13_1,0_2)$	$(0_1,5_1,10_1,14_2)$		
$1^{18}3^1$	$(1_1,0_2,7_2,8_2)$	$(1_1,3_1,7_1,11_2)$	$(1_1,4_2,17_2,19_2)$	$(0_1,13_1,6_2,8_2)$	$(0_1,19_1,9_2,11_2)$	$(0_1,15_1,16_2,17_2)$
	$(0_1,1_1,9_1,18_1)$	$(0_1,20_1,4_2,15_2)$	$(0_1,16_1,10_2,20_2)$	$(0_1,2_2,5_2,14_2)$	$(0_1,8_1,11_1,12_1)$	$(1_1,13_2,16_2,18_2)$
	$(0_1,3_2,7_2,13_2)$	$(1_1,19_1,6_2,10_2)$	$(1_1,14_1,15_2,20_2)$			
$1^{24}3^1$	$(1_1,5_1,0_2,6_2)$	$(0_2,1_2,3_2,10_2)$	$(0_1,3_1,16_2,18_2)$	$(1_1,11_1,21_2,8_2)$	$(1_1,7_1,12_1,19_2)$	$(1_2,6_2,15_2,24_2)$
	$(0_1,3_2,4_2,11_2)$	$(1_1,4_2,9_2,15_2)$	$(0_1,6_1,12_2,20_2)$	$(1_1,26_1,7_2,10_2)$	$(1_1,8_1,18_2,26_2)$	$(0_1,25_1,15_2,19_2)$
	$(0_1,4_1,5_1,21_2)$	$(0_1,24_1,22_2,2_2)$	$(0_1,9_1,11_1,13_2)$	$(1_1,4_1,12_2,25_2)$	$(1_1,9_1,10_1,25_1)$	$(1_1,18_1,23_2,24_2)$
	$(0_1,8_1,10_1,9_2)$					
$1^{30}3^1$	$(1_1,5_1,6_1,7_1)$	$(0_1,5_2,6_2,13_2)$	$(0_1,9_1,16_1,28_2)$	$(1_1,32_1,5_2,26_2)$	$(0_1,15_1,25_2,31_2)$	$(1_1,10_1,18_1,28_1)$
	$(0_1,1_2,3_2,27_2)$	$(1_1,0_2,4_2,12_2)$	$(1_1,13_1,7_2,14_2)$	$(1_1,4_1,11_1,23_2)$	$(0_1,16_2,17_2,29_2)$	$(1_1,10_2,16_2,21_2)$
	$(0_1,2_1,32_1,9_2)$	$(0_1,11_1,30_1,8_2)$	$(1_1,23_1,9_2,19_2)$	$(1_1,8_2,24_2,32_2)$	$(0_1,24_1,15_2,20_2)$	$(1_1,15_2,29_2,31_2)$
	$(0_1,2_2,4_2,23_2)$	$(0_1,5_1,19_1,10_2)$	$(1_1,30_1,3_2,18_2)$	$(0_1,11_2,14_2,24_2)$	$(0_1,26_1,18_2,30_2)$	
$1^{36}3^1$	$(1_1,6_1,7_1,4_2)$	$(0_1,5_1,11_2,26_2)$	$(1_1,31_2,8_2,14_2)$	$(0_1,12_1,16_2,20_2)$	$(0_1,36_1,30_2,10_2)$	$(1_1,33_2,13_2,21_2)$
	$(0_1,2_1,9_1,14_2)$	$(0_1,8_1,11_1,36_2)$	$(1_1,3_1,36_1,29_2)$	$(0_1,13_2,15_2,24_2)$	$(0_2,13_2,14_2,37_2)$	$(1_1,37_1,16_2,23_2)$
	$(1_1,4_1,25_2,3_2)$	$(1_1,11_1,28_2,0_2)$	$(1_1,5_1,14_1,19_2)$	$(0_1,14_1,38_1,23_2)$	$(0_2,19_2,24_2,26_2)$	$(0_1,29_2,33_2,37_2)$
	$(0_1,10_1,17_2,6_2)$	$(1_1,15_1,2_2,36_2)$	$(1_1,5_2,32_2,35_2)$	$(0_1,16_1,18_2,19_2)$	$(1_1,13_1,20_1,24_1)$	$(1_1,16_1,22_1,11_2)$
	$(0_1,22_2,27_2,1_2)$	$(1_1,2_1,21_1,30_2)$	$(1_1,9_1,20_2,17_2)$			
$1^{42}3^1$	$(0_1,30_2,1_2,8_2)$	$(0_1,5_1,14_2,44_2)$	$(1_1,31_1,9_2,11_2)$	$(0_1,13_1,14_1,25_2)$	$(1_1,10_1,12_1,15_2)$	$(1_1,9_1,22_1,44_2)$
	$(0_1,7_1,9_1,43_1)$	$(0_1,6_1,23_2,37_2)$	$(1_1,32_2,40_2,4_2)$	$(0_1,19_2,38_2,43_2)$	$(1_1,11_1,26_2,30_2)$	$(0_1,10_2,39_2,40_2)$
	$(0_2,5_2,9_2,42_1)$	$(0_1,8_1,35_2,15_2)$	$(1_1,4_1,25_2,36_2)$	$(0_1,22_1,26_1,37_1)$	$(1_1,14_2,33_2,38_2)$	$(0_1,44_1,18_2,20_2)$
	$(1_1,41_2,5_2,8_2)$	$(0_2,1_2,17_2,44_2)$	$(1_1,8_1,18_1,24_2)$	$(0_1,24_1,33_2,26_2)$	$(1_1,16_1,44_1,29_2)$	$(1_2,11_2,32_2,35_2)$
	$(0_1,1_1,12_2,28_2)$	$(1_1,15_1,19_1,7_2)$	$(1_1,20_1,37_1,0_2)$	$(0_1,30_1,24_2,34_2)$	$(1_1,17_1,34_2,19_2)$	$(1_1,21_1,37_2,22_2)$
	$(0_1,3_1,29_2,41_2)$					

4.4, 4.6 – 4.8. Adjoin five ideal points, fill in the groups of size g together with the ideal points with GDCs of type $2^{\frac{g}{2}}5^1$ from Lemma 6.3, and fill in the group of size m together with the ideal points with optimal codes of length $m+5$ from Lemma 6.2. The results are optimal codes of length $gt+m+5$. ■

Combining Lemmas 6.1, 6.2, and 6.4, we have the following result.

Theorem 6.1: $A_3(n,5,4) = U(n,3)$ for all integer $n \equiv 5 \pmod 6$, $n \geq 11$.

VII. CASE OF LENGTH $n \equiv 3 \pmod 6$

In this section, we determine the value of $A_3(n,5,4)$ for $n \equiv 3 \pmod 6$.

Lemma 7.1: $A_3(15,5,4) = U(15,3)$.

Proof: The required code is constructed on $\mathbb{Z}_{15}$ with 67 codewords, which are composed of two parts. The first part contains the two codewords $(0_1,3_2,6_1,12_1)$ and $(0_2,6_2,9_1,12_2)$. The other part consists of 65 codewords as follows.

Each of the following 13 codewords will be generated to five codewords by applying the following maps f_s for $s \in [0,4]$ to every element a_i of a codeword, where

$$f_s(a_i) = b_{1+\lfloor j/15 \rfloor} \text{ with}$$
$$b \in [0,14],\ b \equiv a + 6s \pmod{15}, \text{ and}$$
$$j \in [0,29],\ j \equiv a + 15i + 6s \pmod{30}:$$

$(0_1,1_1,4_1,7_2)$ $(3_1,8_2,11_2,0_1)$ $(3_1,13_2,4_1,14_2)$
$(3_1,4_2,2_1,1_2)$ $(4_1,6_1,11_1,8_2)$ $(4_1,12_2,2_1,14_1)$
$(0_1,10_1,6_2,2_1)$ $(0_1,10_2,12_2,7_1)$ $(5_1,13_1,14_2,6_2)$
$(1_1,2_1,5_1,10_2)$ $(0_1,9_1,11_1,13_1)$ $(2_1,11_1,12_1,13_2)$
$(2_1,3_2,7_1,11_2)$.

■

Lemma 7.2: There exists a GDC of type $1^{42}15^1$ and size 987.

Proof: Let $X = \{0,1,\ldots,56\}$, and $\mathcal{G} = \{\{i\} : 0 \leq i \leq 41\} \cup \{\{42,43,\ldots,56\}\}$. Then, $(X,\mathcal{G},\mathcal{C})$ is a GDC of type $1^{42}15^1$ and size 987, where $\mathcal{C}$ is generated from the codewords listed in Table IX, which are developed under the automorphism group $\langle (0\ 2\ 4\ \cdots\ 40)(1\ 3\ 5\ \cdots\ 41)(42\ 43\ 44)(45\ 46\ 47)(48\ 49\ 50)(51\ 52\ 53)(54\ 55\ 56) \rangle$. ■

Lemma 7.3: There exists a GDC of type $1^{n-3}3^1$ and size $\frac{(n-3)(2n+3)}{6}$, and therefore, $A_3(n,5,4) \geq U(n,3) - 1$, for each $n \in \{15,21,27,33,39,45\}$.

Proof: For each $n \in \{15,21,27,33,39,45\}$, let $X_n = \{0,1,\ldots,n-1\}$, and $\mathcal{G}_n = \{\{i\} : 0 \leq i \leq n-4\} \cup \{\{n-3, n-2, n-1\}\}$. Then, $(X_n, \mathcal{G}_n, \mathcal{C}_n)$ is a GDC of type $1^{n-3}3^1$ and size $\frac{(n-3)(2n+3)}{6}$, where $\mathcal{C}_n$ is generated from the codewords listed in Table X, which are developed under the automorphism group G as indicated next.

For $n = 15$, $G = \langle (0\ 4\ 8)(1\ 5\ 9)(2\ 6\ 10)(3\ 7\ 11)(12\ 13\ 14) \rangle$.

For $n \in \{21,27,33,39,45\}$, $G = \langle (0\ 2\ 4 \ldots n-5)(1\ 3\ 5 \ldots n-4)(n-3\ n-2\ n-1) \rangle$. ■

Lemma 7.4: $A_3(n,5,4) = U(n,3)$ for all $n \equiv 3 \pmod{12}$ and $n \geq 51$.

Proof: Take a GDC of type 12^u from Lemma 3.6, adjoin three ideal points, fill in the first $u-1$ groups together with the ideal points with GDCs of type $1^{12}3^1$, and fill in the last group together with the ideal points with an optimal $(15,5,4)_3$ code to obtain the required codes. ■

Lemma 7.5: $A_3(n,5,4) = U(n,3)$ for all $n \equiv 9 \pmod{12}$ and $n \geq 57$.

Proof: For $n = 57$, take a GDC of type $1^{42}15^1$ from Lemma 7.2, and fill in the group of size 15 with an optimal $(15,5,4)_3$ code to get the desired code.

For each $n \in \{69, 81, 93, 105, 117, 129\}$, take a GDC of type $12^u 18^1$ for $4 \le u \le 9$ from Lemma 3.7. Adjoin three ideal points. Fill in one group of size 12 together with the ideal points with an optimal $(15,5,4)_3$ code, and fill in the other groups together with the ideal points with GDCs of types $1^{12}3^1$ or $1^{18}3^1$ to get an optimal $(n,5,4)_3$ code.

For $n = 141$, take a $\{4\}$-GDD of type $2^9 5^1$ (see [18]), and apply the Fundamental Construction with weight 6 to all the points to get a GDC of type $12^9 30^1$. Adjoin three ideal points, fill in one group of size 12 together with the extra points with an optimal $(15,5,4)_3$ code, and fill in the other groups together with the extra points with GDCs of types $1^{12}3^1$ or $1^{30}3^1$ to get the desired code.

For $n = 177$, take a $\{4\}$-GDD of type $2^8 5^1 8^1$ (see [16]), and apply the Fundamental Construction with weight 6 to all the points to get a GDC of type $12^8 30^1 48^1$. Adjoin three ideal points, fill in the group of size 48 together with the three extra points with an optimal $(51,5,4)_3$ code, and fill in the other groups together with the extra points with GDCs of types $1^{12}3^1$ or $1^{30}3^1$ to get the desired code.

For $n = 189$, take a $\mathrm{TD}(7,8)$ from Lemma 2.4. Apply the Fundamental Construction with weight 3 to all the points in the first five groups, weight 3 to 6 points in the sixth group, weight 6 to all the points in the last group, and weight 0 to the remaining points to get a GDC of type $24^5 18^1 48^1$. Here, the input GDCs of types $3^5 6^1$ and $3^6 6^1$ are from Lemmas 2.6 and 3.4. Adjoin three ideal points, fill in the group of size 48 together with the extra points with an optimal $(51,5,4)_3$ code, and fill in the other groups together with the extra points with GDCs of types $1^{24}3^1$ or $1^{18}3^1$ to get the desired code.

For $n \in \{153, 165\}$ or $n \ge 201$, take a $\mathrm{TD}(6,t)$ from Lemma 2.4. Apply the Fundamental Construction with weight 6 to all points in the first four groups, x points in the fifth group, two points in the last group, and weight 3 to another point in the last group. The other points are given weight 0. The result is a GDC of type $(6t)^4(6x)^1 15^1$ for $x = 0$ or $3 \le x \le t$. Fill in each group of the GDC with suitable optimal codes of lengths $6t$, $6x$, or 15 from Theorem 4.1 and Lemma 7.1 to get optimal codes of length $6(4t+x)+15$, where $4t+x$ can take 23, 25, or any odd integer no less than 31. ■

Combining the aforementioned lemmas, we have the following theorem.

Theorem 7.1: $A_3(n,5,4) = U(n,3)$ for $n = 15$ and all $n \equiv 3 \pmod 6$, $n \ge 51$; $A_3(n,5,4) \ge U(n,3) - 1$ for each $n \in \{21, 27, 33, 39, 45\}$.

VIII. Conclusion

In this paper, we determine almost completely the spectrum of sizes for optimal ternary CWCs of weight 4 and distance 5. We summarize our main results of this paper as follows.

Theorem 8.1: For any integer $n \ge 4$,

$$A_3(n,5,4) = \begin{cases} 1, & \text{if } n = 4 \\ 2, & \text{if } n = 5 \\ 4, & \text{if } n = 6 \\ 7, & \text{if } n = 7 \\ 13, & \text{if } n = 8 \\ 19, & \text{if } n = 9 \\ \left\lfloor \frac{n}{2} \left\lfloor \frac{2(n-1)}{3} \right\rfloor \right\rfloor, & \text{if } n \ge 10 \text{ and } n \notin \{12, 13, \\ & 21, 27, 33, 39, 45, 52\}, \end{cases}$$

$$A_3(n,5,4) \in \left[\left\lfloor \frac{n}{2} \left\lfloor \frac{2(n-1)}{3} \right\rfloor \right\rfloor - 1, \left\lfloor \frac{n}{2} \left\lfloor \frac{2(n-1)}{3} \right\rfloor \right\rfloor \right]$$

for each $n \in \{12, 21, 27, 33, 39, 45\}$, $A_3(13,5,4) \in [48, 52]$, and $A_3(52,5,4) \in [872, 884]$.

Acknowledgment

The authors would like to thank Dr. M. Blaum, the Associate Editor of the IEEE Transactions on Information Theory, and the anonymous referee for their constructive comments and suggestions that greatly improved the readability of this article. The authors express their gratitude to Prof. L. Ji for the helpful discussion on the construction method for the optimal $(15,5,4)_3$ code.

References

[1] R. J. R. Abel, C. J. Colbourn, and J. H. Dinitz, "Mutually orthogonal latin squares (MOLS)," in *Handbook of Combinatorial Designs*, C. J. Colbourn and J. H. Dinitz, Eds., 2nd ed. Boca Raton, FL: CRC Press, 2007, pp. 160–193.

[2] S. Blake-Wilson and K. T. Phelps, "Constant weight codes and group divisible designs," *Des. Codes Cryptogr.*, vol. 16, no. 1, pp. 11–27, 1999.

[3] G. Bogdanova, "New bounds for the maximum size of ternary constant weight codes," *Serdica Math. J.*, vol. 26, no. 1, pp. 5–12, 2000.

[4] H. Cao, L. Ji, and L. Zhu, "Constructions for generalized Steiner systems $\mathrm{GS}(3,4,v,2)$," *Des. Codes Cryptogr.*, vol. 45, no. 2, pp. 185–197, 2007.

[5] Y. M. Chee, S. H. Dau, A. C. H. Ling, and S. Ling, "The sizes of optimal q-ary codes of weight three and distance four: A complete solution," *IEEE Trans. Inf. Theory*, vol. 54, no. 3, pp. 1291–1295, Mar. 2008.

[6] Y. M. Chee, G. Ge, and A. C. H. Ling, "Group divisible codes and their application in the construction of optimal constant-composition codes of weight three," *IEEE Trans. Inf. Theory*, vol. 54, no. 8, pp. 3552–3564, Aug. 2008.

[7] Y. M. Chee and S. Ling, "Constructions for q-ary constant-weight codes," *IEEE Trans. Inf. Theory*, vol. 53, no. 1, pp. 135–146, Jan. 2007.

[8] K. Chen, G. Ge, and L. Zhu, "Generalized Steiner triple systems with group size five," *J. Combin. Des.*, vol. 7, no. 6, pp. 441–452, 1999.

[9] D. J. Costello and G. D. Forney, "Channel coding: The road to channel capacity," *Proc. IEEE*, vol. 95, no. 6, pp. 1150–1177, Jun. 2007.

[10] T. Etzion, "Optimal constant weight codes over Z_k and generalized designs," *Discrete Math.*, vol. 169, no. 1-3, pp. 55–82, 1997.

[11] F.-W. Fu, T. Kløve, Y. Luo, and V. K. Wei, "On the Svanström bound for ternary constant-weight codes," *IEEE Trans. Inf. Theory*, vol. 47, no. 5, pp. 2061–2064, Nov. 2001.

[12] F.-W. Fu, A. J. H. Vinck, and S.-Y. Shen, "On the constructions of constant-weight codes," *IEEE Trans. Inf. Theory*, vol. 44, no. 1, pp. 328–333, Jan. 1998.

[13] G. Ge, "Generalized Steiner triple systems with group size $g \equiv 1, 5 \pmod 6$," *Australas. J. Combin.*, vol. 21, pp. 37–47, 2000.

[14] G. Ge, "Further results on the existence of generalized Steiner triple systems with group size $g \equiv 1, 5 \pmod 6$," *Australas. J. Combin.*, vol. 25, pp. 19–27, 2002.
[15] G. Ge, "Generalized Steiner triple systems with group size $g \equiv 0, 3 \pmod 6$," *Acta Math. Appl. Sin. Engl. Ser.*, vol. 18, no. 4, pp. 561–568, 2002.
[16] G. Ge, "Group divisible designs," in *Handbook of Combinatorial Designs*, C. J. Colbourn and J. H. Dinitz, Eds., 2nd ed. Boca Raton, FL: CRC Press, 2007, pp. 255–260.
[17] G. Ge, "Construction of optimal ternary constant weight codes via Bhaskar Rao designs," *Discrete Math.*, vol. 308, no. 13, pp. 2704–2708, 2008.
[18] G. Ge and A. C. H. Ling, "Group divisible designs with block size four and group type $g^u m^1$ with small g," *Discrete Math.*, vol. 285, no. 1-3, pp. 97–120, 2004.
[19] G. Ge and A. C. H. Ling, "Asymptotic results on the existence of 4-RGDDs and uniform 5-GDDs," *J. Combin. Des.*, vol. 13, no. 3, pp. 222–237, 2005.
[20] G. Ge and R. S. Rees, "On group-divisible designs with block size four and group-type $6^u m^1$," *Discrete Math.*, vol. 279, no. 1-3, pp. 247–265, 2004.
[21] G. Ge and D. Wu, "Generalized Steiner triple systems with group size ten," *J. Math. Res. Expo.*, vol. 23, no. 3, pp. 391–396, 2003.
[22] G. Ge and D. Wu, "Some new optimal quaternary constant weight codes," *Sci. China Ser. F*, vol. 48, no. 2, pp. 192–200, 2005.
[23] L. Ji, D. Wu, and L. Zhu, "Existence of generalized Steiner systems $\mathrm{GS}(2, 4, v, 2)$," *Des. Codes Cryptogr.*, vol. 36, no. 1, pp. 83–99, 2005.
[24] O. D. King, "Bounds for DNA codes with constant GC-content," *Electron. J. Combin.*, vol. 10, no. 1, 2003.
[25] D. S. Krotov, "Inductive constructions of perfect ternary constant-weight codes with distance 3," *Proble. Pered. Inf.*, vol. 37, no. 1, pp. 3–11, 2001.
[26] F. J. MacWilliams and N. J. A. Sloane*, The Theory of Error Correcting Codes. II*. Amsterdam, The Netherlands: North-Holland, 1977, vol. 16.
[27] O. Milenkovic and N. Kashyap*, On the Design of Codes for DNA Computing*. Berlin, Germany: Springer-Verlag, 2006, vol. 3969, Lecture Notes in Computer Science, pp. 100–119.
[28] R. C. Mullin and H.-D. O. F. Gronau, "PBDs and GDDs: The basics," in *Handbook of Combinatorial Designs*, C. J. Colbourn and J. H. Dinitz, Eds., 2nd ed. Boca Raton, FL: CRC Press, 2007, pp. 231–236.
[29] P. R. J. Östergård and M. Svanström, "Ternary constant weight codes," *Electron. J. Combin.*, vol. 9, no. 1, 2002.
[30] K. Phelps and C. Yin, "Generalized Steiner systems with block size three and group size $g \equiv 3 \pmod 6$," *J. Combin. Des.*, vol. 5, no. 6, pp. 417–432, 1997.
[31] K. Phelps and C. Yin, "Generalized Steiner systems with block size three and group size four," *Ars Combin.*, vol. 53, pp. 133–146, 1999.
[32] M. Svanström, "A lower bound for ternary constant weight codes," *IEEE Trans. Inf. Theory*, vol. 43, no. 5, pp. 1630–1632, Sep. 1997.
[33] M. Svanström, "A class of perfect ternary constant-weight codes," *Des. Codes Cryptogr.*, vol. 18, no. 1-3, pp. 223–229, 1999.
[34] M. Svanström, "Ternary codes with weight constraints," Ph.D. dissertation, Linköpings Univ., Linköping, Sweden, 1999.
[35] M. Svanström, P. R. J. Östergård, and G. T. Bogdanova, "Bounds and constructions for ternary constant-composition codes," *IEEE Trans. Inform. Theory*, vol. 48, no. 1, pp. 101–111, Jan. 2002.
[36] J. van Lint and L. Tolhuizen, "On perfect ternary constant weight codes," *Des. Codes Cryptogr.*, vol. 18, no. 1-3, pp. 231–234, 1999.
[37] D. Wu, G. Ge, and L. Zhu, "Generalized Steiner triple systems with group size $g = 7, 8$," *Ars Combin.*, vol. 57, pp. 175–191, 2000.
[38] D. Wu and L. Zhu, "Generalized Steiner systems $\mathrm{GS}(2, 4, v, 2)$ with v a prime power $7 \pmod{12}$," *Des. Codes Cryptogr.*, vol. 24, no. 1, pp. 69–80, 2001.
[39] J. Yin, Y. Lu, and J. Wang, "Maximum distance holey packings and related codes," *Sci. China Ser. A*, vol. 42, no. 12, pp. 1262–1269, 1999.
[40] H. Zhang and G. Ge, "Optimal ternary constant-weight codes of weight four and distance six," *IEEE Trans. Inf. Theory*, vol. 56, no. 5, pp. 2188–2203, May 2010.

Hui Zhang is currently a Ph.D. student at Zhejiang University, Hangzhou, Zhejiang, China. Her research interests include combinatorial design theory, coding theory, cryptography, and their interactions.

Xiande Zhang received the Ph.D. degree in mathematics from Zhejiang University, Hangzhou, Zhejiang, China, in 2009. During 2009–2011, she held a postdoctoral position with Mathematical Sciences, Nanyang Technological University, Singapore. She is now a research fellow with School of Mathematical Sciences, Monash University, Australia. Her research interests include combinatorial design theory, coding theory, cryptography, and their interactions.

Gennian Ge received the M.S. and Ph.D. degrees in mathematics from Suzhou University, Suzhou, Jiangsu, China, in 1993 and 1996, respectively. After that, he became a member of Suzhou University. He was a postdoctoral fellow in the Department of Computer Science at Concordia University, Montreal, QC, Canada, from September 2001 to August 2002, and a visiting assistant professor in the Department of Computer Science at the University of Vermont, Burlington, Vermont, USA, from September 2002 to February 2004. Since then, he has been a full professor in the Department of Mathematics at Zhejiang University, Hangzhou, Zhejiang, China. His research interests include the constructions of combinatorial designs and their applications to codes and crypts.

Dr. Ge is on the Editorial Board of the *Journal of Combinatorial Designs*, *The Open Mathematics Journal*, and the *International Journal of Combinatorics*. He received the 2006 Hall Medal from the Institute of Combinatorics and Its Applications.